Business Maths and Statistics

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Preface

This book has grown out of several years of teaching mathematics and statistics for professional bodies examinations to business studies students. It is especially intended for students who do not wish to specialise in these subjects, but to competently understand them, not just in order to pass an examination, but also to see how these skills may be of practical help to them in the business environment. The passing of examinations is of course important and therefore the questions at the end of each chapter are taken from past examination papers.

The book is intended for students studying Statistics or Mathematics and Statistics in the Foundation examinations of the Association of Certified Accountants, the Institute of Chartered Accountants and the Institute of Cost and Management Accountants. It is also useful for students studying for the Institute of Marketing, the Institute of Transport and Statistics in such courses as the BEC national level core and option modules.

The book includes chapters on those topics of mathematics which are particularly pertinent to application in business. This I hope will be of help to those students whose mathematics is rusty and also to those whose mathematical education to date has covered only modern mathematics and who have not met such topics as compound interest and depreciation.

I must express my thanks to all my students on whom I have tried out all the examples used in the book. Also may I thank all my friends and especially my husband for all their encouragement during the writing of this book.

Margaret Tilley

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1 Series

Arithmetic Series

What is the next term in each of the following series?

1. 1, 2, 3, 4, . . .
2. 2, 5, 8, 11, . . .
3. 4, 3, 2, 1, . . .

In each case we progress through the series by adding (or subtracting) the same quantity, so in case 2. we simply add three to obtain the next term. In general, if the first term is a and the quantity to be added each time is d,

the second term is $a + d$

the third term is $a + d + d$, that is $a + 2d$

the fourth term is $a + 2d + d$, that is $a + 3d$

In the three cases we have already looked at

1. $a = 1,\ d = 1$
2. $a = 2,\ d = 3$
3. $a = 4,\ d = -1$ (this has the effect of subtracting).

If we want to know the 10th term for case 2. we could work out the 10 terms, i.e. 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, so the 10th term is 29. Or we could say the 10th term is $a + (n - 1)d$ where $a = 2$, $d = 3$, $n = 10$, so

$$\begin{aligned} a + (n - 1)d &= 2 + (10 - 1) \times 3 \\ &= 2 + 9 \times 3 \\ &= 2 + 27 \\ &= 29 \end{aligned}$$

There is not much difference in the work involved either way for low numbers, but what if we needed to know the 1000th term? On the basis of the above we can generalize about arithmetic series as follows:

if a = first term

and d = common difference (positive or negative)

then general term or nth term is $a + (n - 1)d$.

Sum of an Arithmetic Series

What is the sum of the first n numbers? That is, what is

$$1 + 2 + 3 + 4 + \ldots + n$$

As we will often require to find the sum of the terms in an arithmetic series, let us look at the problem in general terms.

We use S_n to denote the sum of the first n terms of an arithmetic series. So we have

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d] \qquad (1.1)$$

If we write this down again (but backwards), we get

$$S_n = a + (n - 1)d + [a + (n - 2)d] + [a + (n - 3)d] + [a + (n - 4)d] + \ldots + a \qquad (1.2)$$

If we now add equations (1.1) and (1.2) together, we have

$$S_n + S_n = \{ a + [a + (n - 1)d] \} + \{ (a + d) + [a + (n - 2)d] \} + \ldots + \{ [a + (n - 1)d] + a] \}$$

The value of the terms within each curly bracket works out to be $2a + (n - 1)d$. So we have

$$2S_n = n[2a + (n - 1)d]$$

or

$$S_n = \frac{n}{2} [2a + (n - 1)d]$$

If we want the sum of the first 50 whole numbers, 1, 2, 3, . . . , 50, we have $a = 1$, $d = 1$, $n = 50$ and so

$$S_{50} = \tfrac{50}{2} [2 \times 1 + (50 - 1) \times 1]$$
$$= 25 \times 51$$
$$= 1275$$

Example 1.1
Find the 40th term in the arithmetic series $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$.

We have
$$a = \tfrac{1}{4}$$
$$d = \tfrac{1}{4}$$
$$n = 40$$

nth term is $\quad a + (n - 1)d$

$\therefore$ 40th term is $\quad \frac{1}{4} + (40 - 1) \times \frac{1}{4}$

$$= 10$$

Example 1.2
Find the sum of the first 18 terms of the arithmetic series 3, 7, 11, 15, . . .

$$a = 3$$
$$d = 4$$
$$n = 18$$

$$S_n = \frac{n}{2} [2a + (n - 1)d]$$

$$= \frac{18}{2}\,[2 \times 3 + (18 - 1)4]$$
$$= 9\,(6 + 68)$$
$$= 666$$

Geometric Series

This is a series where every item is connected to the previous term by multiplying by a constant, called the common ratio, e.g.

$$1, 2, 4, 8, 16, \ldots$$
$$3, 1\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{8}, \ldots$$
$$1, -3, 9, -27, \ldots$$

In general, using the notation

$$a = \text{first term}$$
$$r = \text{common ratio}$$

the series is

$$a, ar, ar^2, ar^3 \ \ldots$$

Notice that the third term is ar^2

fourth term is ar^3

and in general the

nth term is ar^{n-1}

Thus the first term is $ar^{1-1} = ar^0 = a$. (Remember $r^0 = 1$).

Again we will require the sum of the first n terms of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \qquad (1.3)$$

Multiplying equation (1.3) by r and moving each term along by one, gives

$$rS_n = \quad ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad (1.4)$$

Then (1.3) minus (1.4) gives

$$S_n - rS_n = a - ar^n$$

or

$$S_n(1 - r) = a(1 - r^n)$$

or

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

We use this arrangement of the formula if $r < 1$. If $r > 1$ it is more convenient to use the formula as

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Example 1.3
Find the 10th term and the sum of the first 10 terms of the series given by 1, 2, 4, ...

Here $a = 1$

$r = 2$

$$n\text{th term} = ar^{n-1}$$
$$= 1 \times 2^9 = 512$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{where } n = 10$$

$$S_{10} = \frac{1(2^{10} - 1)}{2 - 1}$$
$$= 2^{10} - 1$$
$$= 1024 - 1 = 1023$$

Example 1.4
Find the sum of the first eight terms of the series 16, –8, 4, ...

Here $a = 16$

$r = -\frac{1}{2}$

$n = 8$

In this case it is more convenient to use the formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_8 = \frac{16[1 - (-\frac{1}{2})^8]}{1 - (-\frac{1}{2})} \qquad \left((-\tfrac{1}{2})^8 = \frac{(-1)^8}{2^8} = \frac{1}{256}\right)$$

$$= \frac{16\left(1 - \frac{1}{256}\right)}{1 + \frac{1}{2}}$$

$$= \frac{16 \times \frac{255}{256}}{\frac{3}{2}}$$

$$= 16 \times \tfrac{255}{256} \times \tfrac{2}{3}$$

$$= 10{\cdot}625$$

Compound Interest and Discounting Formulae

The work we have done on series will be needed to help us deal with problems of interest and depreciation.

Simple Interest

If we invest £P at R% per annum for T years, then the interest I is given by

$$I = \frac{PRT}{100}$$

So £100 invested at 4% for 10 years will earn

$$\frac{100 \times 4 \times 10}{100} = £40$$

Usually the money earned each year is reinvested giving a greater principal for the second and successive years. We can use the ideas of geometric series to save the very tedious arithmetic in calculating compound interest.

Compound Interest

Suppose we invest £P at R% per annum compound interest for n years.

$$\text{Value at the end of the first year} = P + \frac{PR}{100}$$

$$= P\left(1 + \frac{R}{100}\right)$$

$$\text{value at the end of the second year} = P\left(1 + \frac{R}{100}\right)$$

$$+ P\left(1 + \frac{R}{100}\right)\frac{R}{100}$$

$$= P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right)$$

$$\text{or } P\left(1 + \frac{R}{100}\right)^2$$

If we do similarly for successive years, this gives:

$$\text{value at the end of } n \text{ years} = P\left(1 + \frac{R}{100}\right)^n$$

Example 1.5
Calculate the value after 10 years of £300 invested at 9% compound interest per annum.

$$P = 300$$

$$R = 9$$

$$n = 10$$

$$\text{Value at end of } n \text{ years} = P\left(1 + \frac{R}{100}\right)^n$$

$$\text{Value at end of 10 years} = 300\left(1 + \frac{9}{100}\right)^{10}$$

$$= 300 \times 1{\cdot}09^{10}$$

$$= 300 \times 2{\cdot}367$$

$$= 710{\cdot}1$$

$$= £710 \text{ (to the nearest pound)}$$

Discounting Formula

We can use the same method as that for compound interest when depreciation is at a constant percentage per annum, as in the following example.

Example 1.6
A piece of machinery is purchased for £20 000. If it depreciates at 10% per annum, calculate the value of the machine at the end of 20 years.

We can use the same method as for Example 1.5, except that

$$R = -10$$

$$P = £20\,000$$

$$n = 20$$

$$\text{Value at end of } n \text{ years} = P\left(1 + \frac{R}{100}\right)^n$$

$$\text{Value at end of 20 years} = 20\,000\left(1 + \frac{-10}{100}\right)^{20}$$

$$= 20\,000\,(1 - 0{\cdot}1)^{20}$$

$$= 20\,000 \times 0{\cdot}9^{20}$$

$$= 20\,000 \times 0{\cdot}1216$$

$$= £2432$$

Approximate value of machine after 20 years is £2400 (to the nearest £100).

Stepped-payment Problems

We must also consider problems where the sum invested or owed changes at certain time intervals.

Example 1.7

A man invests £10 per month in a Building Society Savings Account for 5 years. The interest is added to his account monthly; the rate is 7·5% per annum. Calculate the value of the account at the end of 5 years.

$$\text{Rate of interest per month} = \frac{7{\cdot}5\%}{12}$$

$$\text{or } 0{\cdot}625\%$$

$$\text{Value at the beginning of the first month} = 10$$

$$\text{Value at the end of the first month} = 10 + 10 \times \frac{0{\cdot}625}{100}$$

$$= 10\left(1 + \frac{0{\cdot}625}{100}\right)$$

$$= 10\,(1{\cdot}00625)$$

Value at the beginning of the second month $= 10\,(1{\cdot}00625) + 10$

Value at the end of the second month $= 10\,(1{\cdot}00625)^2 + 10\,(1{\cdot}00625)$

Similarly, value at the end of 5 years or 60 months

$$= 10(1{\cdot}00625)^{60} + 10(1{\cdot}00625)^{59} + 10(1{\cdot}00625)^{58} + \ldots$$
$$+ 10(1{\cdot}00625)^2 + 10(1{\cdot}00625)$$

The terms could also be arrived at by considering that the first £10 is invested at compound interest of 0·625% per month for 60 months. Similarly, the second £10 is invested for 59 months, but the last £10 is only invested for one month. We can find an expression for the value at the end of 5 years using the sum S_n of a geometric series where

$$a = 10 \times 1{\cdot}00625$$

$$n = 60$$

$$r = 1{\cdot}00625$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{60} = \frac{10 \times 1{\cdot}00625 \times (1{\cdot}00625^{60} - 1)}{1{\cdot}00625 - 1}$$

$$= \frac{10 \times 1{\cdot}00625}{0{\cdot}00625} \times (1{\cdot}453 - 1)$$

$$= £729{\cdot}3$$

Value at the end of 5 years is approximately £730.

Example 1.8
£5000 is borrowed to purchase a house. The monthly repayments are £50. The interest is calculated annually at 10% per annum. Calculate the amount outstanding after 10 years.

Amount paid per annum is £(50 × 12) = £600

Amount outstanding at the end of the first year is

$$= 5000\left(1 + \frac{10}{100}\right) - 600$$

$$= 5000 \times 1{\cdot}1 - 600$$

Amount outstanding at the end of the second year is

$$(5000 \times 1{\cdot}1 - 600) \times 1{\cdot}1 - 600$$

$$= 5000 \times 1{\cdot}1^2 - 600 \times 1{\cdot}1 - 600$$

Amount outstanding at the end of 10 years is

$$5000 \times 1{\cdot}1^{10} - 600 \times 1{\cdot}1^9 - 600 \times 1{\cdot}1^8 - \ldots - 600 \times 1{\cdot}1 - 600$$

$$= 5000 \times 1{\cdot}1^{10} - 600\,(1{\cdot}1^9 + 1{\cdot}1^8 + \ldots + 1{\cdot}1 + 1)$$

Calculating $S_n = \dfrac{a(r^n - 1)}{r - 1}$ for $a = 1, r = 1{\cdot}1, n = 10$ gives

$$S_{10} = \frac{(1{\cdot}1^{10} - 1)}{1{\cdot}1 - 1} = \frac{2{\cdot}594 - 1}{0{\cdot}1} = 15{\cdot}94$$

Amount outstanding at the end of 10 years is

$$(5000 \times 2{\cdot}954 - 600 \times 15{\cdot}94)$$

$$= £3406$$

Binomial Expansion

We will require to find the terms in the expansion of $(a + x)^n$ where a and x are any numbers and n is a positive integer.

If $n = 2$ then $(a + x)^2 = a^2 + 2ax + x^2$

If $n = 3$ then $(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

To find the terms for larger n, one way is to use a device known as Pascal's triangle.

Pascal's triangle (see Figure 1.1) is obtained by adding pairs of numbers in one line to arrive at the terms in the next line. Each line starts and ends with a one.

Using Pascal's triangle to find the terms in the expansion of $(a + x)^6$, we find

$$(a + x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$$

Notice the powers of a and x, each term decreases the power of a by one and increases the power of x by one.

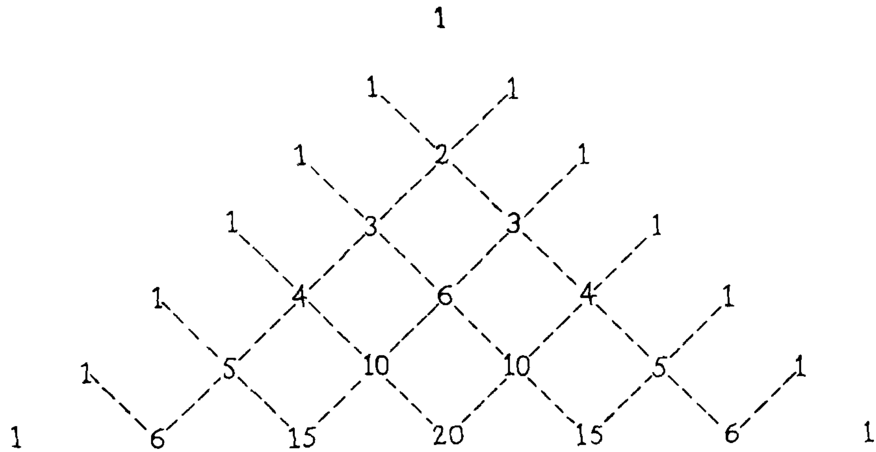

Figure 1.1 Pascal's triangle.

Example 1.9
Find the terms in the expansion of $(2 + x)^5$.

$$\begin{aligned}(2 + x)^5 &= 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^3 \\ &\quad + 5 \times 2x^4 + x^5 \\ &= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5\end{aligned}$$

Pascal's triangle is an easy and suitable method of calculating the coefficients of the Binomial expansion, when n is small, but if n is, say, 30 it would be very tedious to write out all the lines in Pascal's triangle. Consider

$$\begin{aligned}(a + x)^6 &= a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 \\ &\quad + 6ax^5 + x^6 \\ &= a^6 + 6a^5x + \frac{6.1}{2\cdot 1}a^4x^2 + \frac{6.5.4}{3\cdot 2\cdot 1}a^3x^3 \\ &\quad + \frac{6.5.4.3}{4.3.2.1}a^2x^4 + \frac{6.5.4.3.2}{5.4.3.2.1}ax^5 + x^6\end{aligned}$$

Notice the pattern, which we can extend

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2.1}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3\cdot2\cdot1}a^{n-3}x^3 + \ldots x^n$$

The general term in the expansion is, for the rth term,

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots1}a^{n-r}x^r = {}^nC_r\,a^{n-r}x^r$$

$$\text{where } {}^nC_r = \frac{n!}{(n-r)!r!} \qquad \left(\begin{array}{l} n! = n(n-1)(n-2)\ldots3.2.1 \\ \text{(pronounced } n \text{ factorial)}\end{array}\right)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots1}{(n-r)(n-r-1)\ldots1 \times r(r-1)(r-2)\ldots1}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots1}$$

Notice ${}^nC_r = {}^nC_{n-r}$, which gives us the symmetrical coefficients as in Pascal's triangle.

Example 1.10
Find the coefficients of x^3 and x^7 in the expansion of $(2 + x)^{10}$.

The coefficient of x^r in $(a + x)^r$ is ${}^nC_r a^{n-r}$. We have

$$n = 10,\ a = 2,\ r = 3$$

So the coefficient of x^3 is

$${}^{10}C_3 2^7 = \frac{10.9.8}{3.2.1} \times 2^7 = 120 \times 128 = 15\,360$$

The coefficient of x^7 is

$${}^{10}C_7 2^3 = {}^{10}C_3 2^3$$
$$= 120 \times 8 = 960$$

Example 1.11
Use the Binomial expansion to find $1{\cdot}01^5$ correct to four decimal places.

$$1{\cdot}01^5 = (1 + 0{\cdot}01)^5$$
$$= 1 + 5 \times 0{\cdot}01 + 10 \times 0{\cdot}01^2 + 10 \times 0{\cdot}01^3 + 5 \times 0{\cdot}01^4 + 0{\cdot}01^5$$

Notice that each successive term is smaller than the preceding term, so we only need evaluate terms which will affect the fourth decimal place.

$$(1 + 0{\cdot}01)^5 = 1 + 0{\cdot}05 + 0{\cdot}001 + 0{\cdot}00001 + \ldots$$

$$= 1{\cdot}05101$$

$$= 1{\cdot}0510 \quad \text{(correct to four decimal places)}$$

In limited cases we can use the Binomial expansion when n is not a positive integer. We can expand

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2.1}x^2 + \frac{n(n-1)(n-2)}{3.2.1}x^3 + \ldots$$

$$+ \frac{n(n-1)\ldots(n-r+1)}{r(r-1)\ldots 1}x^r + \ldots$$

If $-1 < x < 1$. The expansion contains infinitely many terms when n is not a positive integer.

Example 1.12

Use the Binomial expansion to evaluate $\sqrt{0{\cdot}8}$ to four decimal places.

$$\sqrt{0{\cdot}8} = 0{\cdot}8^{1/2} = (1 - 0{\cdot}2)^{1/2}$$

$$= 1 + \tfrac{1}{2}(-0{\cdot}2) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2{\cdot}1}(-0{\cdot}2)^2$$

$$+ \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3.2.1}(-0{\cdot}2)^3 + \ldots$$

$$= 1 - 0{\cdot}1 - 0{\cdot}005 - 0{\cdot}0005 - 0{\cdot}0000625 \ldots$$

$$= 1 - 0{\cdot}1055625$$

$$= 0{\cdot}8944375$$

$$= 0{\cdot}8944 \text{ (correct to four decimal places)}$$

Exponential Series

The exponential series, or the expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \ldots$$

The value of e can be calculated to any degree of accuracy required using the series

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$= 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \ldots$$

$$= 2{\cdot}7183 \text{ (correct to four decimal places)}$$