NORMAN SCHOFIELD

Mathematical Methods in Economics

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Biographical Details

Norman Schofield obtained B.Sc. degrees in both Physics and Mathematics at Liverpool University and a Ph.D. from Essex University. His principle research interests are in social choice, game theory and mathematical applications in political economy. He has published about thirty articles in these fields in a number of economics, mathematics, political science and sociology journals and edited two books - Crisis in Economic Relations Between North and South (Gower Publishers, Aldershot, England) and Data Analysis and the Social Sciences (Frances Pinter Publishers, London, and St. Martin's Press, New York). He is currently reader in economics at Essex University. During 1982-83 he was Hallsworth Fellow in Political Economy at Manchester University and in 1983-84 was the Sherman Fairchild Distinguished Scholar at the California Institute of Technology. He has previously taught at Yale University and the University of Texas at Austin. He is on the editorial board of the new journal Social Choice and Welfare. His book, Social Choice and Democracy, will be published by Springer Verlag (Heidelberg and New York) in 1984, and he is currently working on books on International Political Economy and General Equilibrium Theory.

FOREWORD

In recent years, the usual optimisation techniques, which have proved so useful in microeconomic theory, have been extended to incorporate more powerful topological and differential methods, and these methods have led to new insights into the qualitative behaviour of general economic systems. These developments have necessarily resulted in an increase in the degree of formalism in the publications in the academic economic theory journals; a formalism which can often deter graduate students. My hope is that the progression of ideas presented here will familiarise the student with the geometric concepts underlying these topological methods, and, as a result, make modern mathematical economics and general equilibrium theory more accessible.

The first chapter of the book introduces the general idea of mathematical structure and representation, while the second chapter analyses linear systems and the representation of transformations of linear systems by matrices. In the third chapter, topological ideas and continuity are introduced, and made use of in solving convex optimisation problems. These procedures then lead naturally to calculus techniques for using a linear approximation, the differential, of a function to study its "local" behaviour.

The book is not intended to cover mathematical economics or general equilibrium theory. However in the last sections of the third and fourth chapters I have introduced some of the standard tools of economic theory, namely the Kuhn Tucker Theorem, some elements of convex analysis and procedures using the Langrangian, and provided examples of consumer and producer optimisation. The final section of chapter four also discusses in a fairly heuristic fashion the smooth or critical Pareto set and the idea of a regular economy. The fifth and final chapter is somewhat more advanced, and extends the differential calculus of a real valued function to the analysis of a smooth function between "local" vector spaces, or manifolds. Modern singularity theory is the study and classification of all such smooth functions, and the purpose of the final chapter has been to use this perspective to obtain a generic or typical picture of the Pareto set and the set of Walrasian equilibria of an exchange economy.

Since the underlying mathematics of this final section are rather difficult, I have not attempted rigorous proofs, but rather sought to lay out the natural path of development from elementary differential calculus to the powerful tools of singularity theory. In the text I have referred to work of Debreu, Balasko, Smale and others who, in the last few years, have used the tools of singularity theory to develop a deeper insight into the geometric structure of an economy. Review exercises are provided at the end of the book, for the use of the reader.

I am indebted to my graduate students for the pertinent questions they asked during the course on mathematical methods in economics, which I gave at Essex University during 1979-1982. It is a pleasure to thank Mike Martin of Essex University and Peter Lambert of the University of York for the helpful suggestions they made, and Pam Hepworth and Nancy Tobbell for typing the manuscript.

I am grateful to the Economics Department of Manchester University for the opportunity provided by a Hallsworth Fellowship in Political Economy in 1982-83 during which I completed the book.

I hope that this book awakens an interest in mathematics for the reader, just as mine was by Terry Wall, Mike Butler, Ian Porteous and Tom Wilmore at Liverpool University.

Pasadena, California
January 1984

CONTENTS

MATHEMATICAL METHODS IN ECONOMICS

Chapter 1.

SETS, RELATIONS AND PREFERENCES

In this chapter we introduce the elementary set theory and notation to be used throughout the book. We also define the notions of binary relation, function, as well as the axioms of a group and field. Finally we discuss the idea of an individual and social preference relation, and mention some of the concepts of social choice and welfare economics.

1.1. ELEMENTS OF SET THEORY

Let U be a collection of objects, which we shall call the domain of discourse, universal set or universe. A set B in this universe (or subset of U) is a subcollection of objects from U. B may be defined either explicitly by enumerating the objects, for example by writing

$$B = \{\text{Tom, Dick, Harry}\}$$
$$\text{or } B = \{x_1, x_2, x_3, \ldots\}.$$

Alternatively B may be defined implicitly be reference to some property P(B), which characterises the elements of B, thus

$$B = \{x: \text{ x satisfies } P(B)\}.$$

For example:

$B = \{x: \text{ x is an integer satisfying } 1 \leq x \leq 5\}$
is a satisfactory definition of the set B, where the universal set could be the collection of all integers. If B is a set write $x \in B$ to mean that the element x is a member of B. Write $\{x\}$ for the set which contains only one element, x.

If A, B are two sets write $A \cap B$ for the set which contains only those elements which are both in A and B, and $A \cup B$ for the set whose elements are either in A or B. The null set Φ, is that subset of U which contains no elements in U.

Finally if A is a subset of U, define the negation of A, or complement of A in U to be the set

$$\bar{A} = \{x: \text{ x is in U but not in A}\}.$$

1.1.1. A Set Theory

Now let T be a family of subsets of U, where T includes both U and Φ, i.e.

$$T = \{U, \Phi, A, B, \dots\}.$$

If A is a member of T, then write $A \in T$. Note here that T is a set of sets.

Suppose that T satisfies the following properties:

i) for any $A \in T$, $\bar{A} \in T$

ii) for any A, B in T, $A \cup B$ is in T

iii) for any A, B in T, $A \cap B$ is in T.

In this case we say that $\mathcal{T}$ satisfies *closure* with respect to $(\overline{}, \cup, \cap)$, and call $\mathcal{T}$ a *set theory*.

For example let 2^U be the set of *all* subsets of U, including both U and Φ. Clearly 2^U satisfies closure with respect to $(\overline{}, \cup, \cap)$.

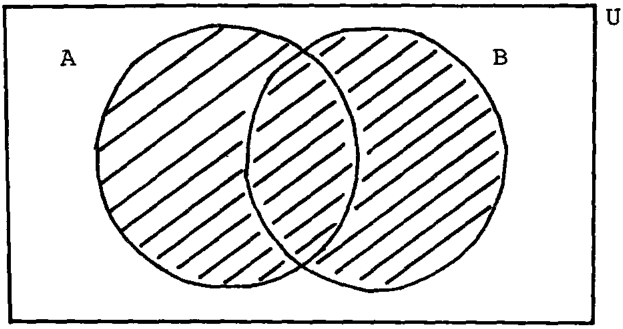
Since a set theory $\mathcal{T}$ satisfies the following axioms we shall call it a *Boolean algebra*.

Axioms

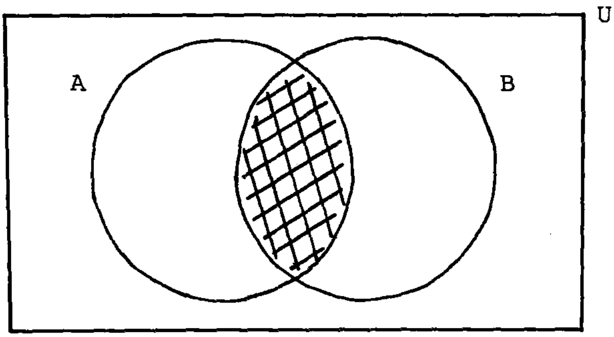
S1.	Zero element	$A \cup \Phi = A$, $A \cap \Phi = \Phi$
S2.	Identity element	$A \cup U = U$, $A \cap U = A$
S3.	Idempotency	$A \cup A = A$, $A \cap A = A$
S4.	Negative	$A \cup \bar{A} = U$, $A \cap \bar{A} = \Phi$ $\bar{\bar{A}} = A$
S5.	Commutativity	$A \cup B = B \cup A$ $A \cap B = B \cap A$
S6.	De Morgan Rule	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$
S7.	Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
S8.	Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

These axioms can be illustrated by Venn diagrams in the following way.

Let the square on the page represent the universal set U. A subset B of points within U can then represent the set B. Given two subsets A, B the union is the hatched area, while the intersection is the double hatched area.



$A \cup B$



$A \cap B$

Fig.1.1.

We shall use $\subset$ to mean "included in". Thus $A \subset B$ means that every element in A is also an element of B.

Thus:

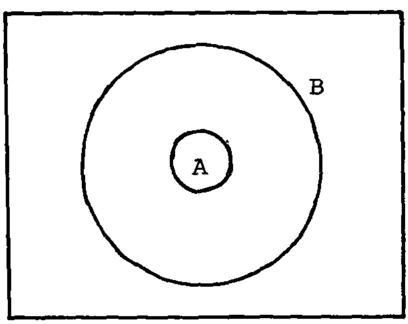


Fig.1.2.

Suppose now that P(A) is the property that characterises A.

Thus $A = \{x: \; x \text{ satisfies } P(A)\}$.

We use the symbol $\equiv$ to mean "identical to", and so $[x \in A] \equiv$ "x satisfies P(A)".

Here "x satisfies P(A)" is a proposition. Associated to any set theory is a <u>propositional calculus</u> which satisfies analogous properties, except that we use $\wedge$ and V instead of the symbols $\cap$ and $\cup$ for "and" and "or".

For example:

$$A \cup B = \{x: \text{ "x satisfies P(A)" V "x satisfies P(B)"}\}$$

$$A \cap B = \{x: \text{ "x satisfies P(A)" } \wedge \text{ "x satisfies P(B)"}\} .$$

The analogue of "$\subset$" is "if ... then" or "implies" which is written =>.

Thus

$$A \subset B \equiv [\text{"x satisfies P(A)"} \Rightarrow \text{"x satisfies P(B)"}]$$

The analogue of "=" in set theory is the symbol "<=>" which means "if and only if", generally written "iff".

For example

$$[A = B] \equiv [\text{"x satisfies P(A)"} \Leftrightarrow \text{"x satisfies P(B)"}]$$

Hence

$$[A = B] \equiv [\text{"}x \in A\text{"} \Leftrightarrow \text{"}x \in B\text{"}]$$

$$\equiv [A \subset B \text{ and } B \subset A]$$

1.1.2. A Propositional Calculus

Let $\{U, \Phi, P_1, \ldots, P_i, \ldots\}$ be a family of simple propositions. U is the universal proposition and always true, whereas Φ is the null proposition and always false. Two propositions P_1, P_2 can be combined to give a proposition $P_1 \wedge P_2$ (i.e. P_1 and P_2) which is true iff both P_1 and P_2 are true, and a proposition $P_1 \vee P_2$ (i.e. P_1 or P_2) which is true if either P_1 or P_2 is true. For a proposition P, the complement $\bar{P}$ in U is true iff P is false, and is false iff P is true.

Now extend the family of simple propositions to a family $\mathcal{P}$, by including in $\mathcal{P}$ any propositional sentence $S(P_1, \ldots, P_i, \ldots)$ which is made up of simple propositions combined under $-, \vee, \wedge$. Then $\mathcal{P}$ satisfies closure with respect to $(-, \vee, \wedge)$ and is called a propositional calculus.

Let T be the truth function, which assigns to any simple proposition, P_i, the value 0 if P_i is false, and 1 if P_i is true. Then T extends to sentences in the obvious way, following the rules of logic, to give a truth function $T: \mathcal{P} \to \{0,1\}$. If $T(S_1) = T(S_2)$ for all truth values of the constituent simple propositions of the sentences S_1 and S_2, then $S_1 = S_2$ (i.e. S_1 and S_2 are identical propositions).

For example the truth values of the proposition $P_1 \vee P_2$ and $P_2 \vee P_1$ are given by the table:

$T(P_1)$	$T(P_2)$	$T(P_1 \vee P_2)$	$T(P_2 \vee P_1)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Since $T(P_1 \vee P_2) = T(P_2 \vee P_1)$ for all truth values it must be the case that $P_1 \vee P_2 = P_2 \vee P_1$.