ZURICH LECTURES IN ADVANCED MATHEMATICS

Pavel Etingof

Calogero-Moser systems and representation theory

European Mathematical Society

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2000 Mathematics Subject Classification 16G30, 70H06, 16S80, 14A22

ISBN 978-3-03719-034-0

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

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European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the author's TEX files: I. Zimmermann, Freiburg
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany
9 8 7 6 5 4 3 2 1

Zurich Lectures in Advanced Mathematics

Edited by

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Published with the support of the Huber-Kudlich-Stiftung, Zürich

To my mother Yelena Etingof on her 75th birthday, with admiration

Contents

Introduction

Calogero–Moser systems, which were originally discovered by specialists in integrable systems, are currently at the crossroads of many areas of mathematics and within the scope of interests of many mathematicians. More specifically, these systems and their generalizations turned out to have intrinsic connections with such fields as algebraic geometry (Hilbert schemes of surfaces), representation theory (double affine Hecke algebras, Lie groups, quantum groups), deformation theory (symplectic reflection algebras), homological algebra (Koszul algebras), Poisson geometry, etc. The goal of the present lecture notes is to give an introduction to the theory of Calogero–Moser systems, highlighting their interplay with these fields. Since these lectures are designed for non-experts, we give short introductions to each of the subjects involved, and provide a number of exercises.

We now describe the contents of the lectures in more detail.

In Lecture 1, we give an introduction to Poisson geometry and to the process of classical Hamiltonian reduction. More specifically, we define Poisson manifolds (smooth, analytic, and algebraic), momemt maps and their main properties, and then describe the procedure of (classical) Hamiltonian reduction. We give an example of computation of Hamiltonian reduction in algebraic geometry (the commuting variety). Finally, we define Hamiltonian reduction along a coadjoint orbit, and give the example which plays a central role in these lectures – the Calogero–Moser space of Kazhdan, Kostant, and Sternberg.

In Lecture 2, we give an introduction to classical Hamiltonian mechanics and the theory of integrable systems. Then we explain how integrable systems may sometimes be constructed using Hamiltonian reduction. After this we define the classical Calogero–Moser integrable system using Hamiltonian reduction along a coadjoint orbit (the Kazhdan–Kostant–Sternberg construction), and find its solutions. Then, by introducing coordinates on the Calogero–Moser space, we write both the system and the solutions explicitly, thus recovering the standard results about the Calogero–Moser system. Finally, we generalize these results to construct the trigonometric Calogero–Moser system.

Lecture 3 is an introduction to deformation theory. This lecture is designed, in particular, to enable us to discuss quantum-mechanical versions of the notions and results of Lectures 1 and 2 in a manner parallel to the classical case. Specifically, we develop the theory of formal and algebraic deformations of assosiative algebras, introduce Hochschild cohomology and discuss its role in studying deformations, and define universal deformations. Then we discuss the basics of the theory of deformation quantization of Poisson (in particular, symplectic) manifolds, and state the Kontsevich quantization theorem.

Lecture 4 is dedicated to the quantum-mechanical generalization of the material of Lecture 1. Specifically, we define the notions of quantum moment map and quantum Hamiltonian reduction. Then we give an example of computation of quantum reduction (the Levasseur–Stafford theorem), which is the quantum analog of the example of commuting variety given in Lecture 1. Finally, we define the notion of quantum reduction with respect to an ideal in the enveloping algebra, which is the quantum version of reduction along a coadjoint orbit, and give an example of this reduction, namely the construction of the spherical subalgebra of the rational Cherednik algebra. Being a quantization of the Calogero–Moser space, this algebra is to play a central role in subsequent lectures.

Lecture 5 contains the quantum-mechanical version of the material of Lecture 1. Namely, after recalling the basics of quantum Hamiltonian mechanics, we introduce the notion of a quantum integrable system. Then we explain how to construct quantum integrable systems by means of quantum reduction (with respect to an ideal), and give an example of this which is central to our exposition: the quantum Calogero–Moser system.

In Lecture 6, we define and study more general classical and quantum Calogero–Moser systems, which are associated to finite Coxeter groups (they were introduced by Olshanetsky and Perelomov). The systems defined in previous lectures correspond to the case of the symmetric group. In general, these integrable systems are not known (or expected) to have a simple construction using reduction; in their construction and study, Dunkl operators are an indispensible tool. We introduce the Dunkl operators (both classical and quantum), and explain how the Olshanetsky–Perelomov Hamiltonians are constructed from them.

Lecture 7 is dedicated to the study of the rational Cherednik algebra, which naturally arises from Dunkl operators (namely, it is generated by Dunkl operators, coordinates, and reflections). Using the Dunkl operator representation, we prove the Poincaré–Birkhoff–Witt theorem for this algebra, and study its spherical subalgebra and center.

In Lecture 8, we consider symplectic reflection algebras, associated to a finite group G of automorphisms of a symplectic vector space V. These algebras are natural generalizations of rational Cherednik algebras (although in general they are not related to any integrable system). It turns out that the PBW theorem does generalize to these algebras, but its proof does not, since Dunkl operators do not have a counterpart. Instead, the proof is based on the theory of deformations of Koszul algebras, due to Drinfeld, Braverman–Gaitsgory, Polishchuk–Positselski, and Beilinson–Ginzburg–Soergel. We also study the spherical subalgebra of the symplectic reflection algebra, and show by deformation-theoretic arguments that it is commutative if the Planck constant is equal to zero.

In Lecture 9, we describe the deformation-theoretic interpretation of symplectic reflection algebras. Namely, we show that they are universal deformations of semidirect products of G with the Weyl algebra of V.

In Lecture 10, we study the center of the symplectic reflection algebra in the case when the Planck constant equals zero. Namely, we consider the spectrum of the center, which is an algebraic variety analogous to the Calogero–Moser space, and show that the smooth locus of this variety is exactly the set of points where the symplectic reflection algebra is an Azumaya algebra; this requires some tools from homological algebra, such as the Cohen–Macaulay property and homological dimension, which we briefly introduce. We also study finite dimensional representations of symplectic reflection algebras with the zero value of the Planck constant. In particular, we show that for G being the symmetric group S_n (i.e. in the case of rational Cherednik algebras of type A), every irreducible representation has dimension $n!$, and irreducible representations are parametrized by the Calogero–Moser space defined in Lecture 1. A similar theorem is valid if $G = S_n \ltimes \Gamma^n$, where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Lecture 11 is dedicated to representation theory of rational Cherednik algebras with a nonzero Planck constant. Namely, by analogy with semisimple Lie algebras, we develop the theory of category $\mathcal{O}$. In particular, we introduce Verma modules, irreducible highest weight modules, which are labeled by representations of G, and compute the characters of the Verma modules. The main challenge is to compute the characters of irreducible modules, and find out which of them are finite dimensional. We do some of this in the case when $G = S_n \ltimes \Gamma^n$, where Γ is a cyclic group. In particular, we construct and compute the characters of all the finite dimensional simple modules in the case $G = S_n$ (rational Cherednik algebra of type A). It turns out that a finite dimensional simple module exists if and only if the parameter k of the Cherednik algebra equals r/n, where r is an integer relatively prime to n. For such values of k, such representation is unique, its dimension is $|r|^{n-1}$, and it has no self-extensions.

At the end of each lecture, we provide remarks and references, designed to put the material of the lecture in a broader prospective, and link it with the existing literature. However, due to a limited size and scope of these lectures, we were, unfortunately, unable to give an exhaustive list of references on Calogero–Moser systems; such a list would have been truly enormous.

Acknowledgments. These lecture notes are dedicated to my mother Yelena Etingof on the occasion of her 75th birthday. Her continuous care from my early childhood to this day has shaped me both as a person and as a mathematician, and there are no words that are sufficient to express my gratitude and admiration.

I gave these lectures in the spring and summer of 2005 at ETH (Zurich). I am greatly indebted to Professor Giovanni Felder and the participants of his seminar for being active listeners; they are responsible for the very existence of these notes, as well as for improvement of their quality. I am also very grateful to ETH for its hospitality and financial support.

This work was partially supported by the NSF grant DMS-0504847 and the CRDF grant RM1-2545-MO-03.

1 Poisson manifolds and Hamiltonian reduction

1.1 Poisson manifolds

Let A be a commutative algebra over a field k.

Definition 1.1. We say that A is a Poisson algebra if it is equipped with a Lie bracket $\{\,,\}$ such that $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

Let I be an ideal in A.

Definition 1.2. We say that I is a Poisson ideal if $\{A, I\} \subset I$.

In this case A/I is a Poisson algebra.
Let M be a smooth manifold.

Definition 1.3. We say that M is a Poisson manifold if its structure algebra $C^\infty(M)$ is equipped with a Poisson bracket.

The same definition can be applied to complex analytic and algebraic varieties: a Poisson structure on them is just a Poisson structure on the structure sheaf. Note that this definition may be used even for singular varieties.

Definition 1.4. A morphism of Poisson manifolds (= Poisson map) is a regular map $M \to N$ that induces a homomorphism of Poisson algebras $C^\infty(N) \to C^\infty(M)$, i.e. a map that preserves Poisson structure.

If M is a smooth variety (C^∞, analytic, or algebraic), then a Poisson structure on M is defined by a Poisson bivector $\Pi \in \Gamma(M, \wedge^2 TM)$ such that its Schouten bracket with itself is zero: $[\Pi, \Pi] = 0$. Namely, $\{f, g\} := (df \otimes dg)(\Pi)$ (the condition that $[\Pi, \Pi] = 0$ is equivalent to the Jacobi identity for $\{\,,\}$). In particular, if M is symplectic (i.e. equipped with a closed nondegenerate 2-form ω) then it is Poisson with $\Pi = \omega^{-1}$, and conversely, a Poisson manifold with nondegenerate Π is symplectic with $\omega = \Pi^{-1}$.

For any Poisson manifold M, we have a homomorphism of Lie algebras $v\colon C^\infty(M) \to \mathrm{Vect}_\Pi(M)$ from the Lie algebra of functions on M to the Lie algebra of vector fields on M preserving the Poisson structure, given by the formula $f \mapsto \{f, ?\}$. In classical mechanics, one says that $v(f)$ is the Hamiltonian vector field corresponding to the Hamiltonian f.

Exercise 1.5. If M is a connected symplectic manifold, then $\mathrm{Ker}(v)$ consists of constant functions. If in addition $H^1(M,\mathbb{C}) = 0$ then the map v is surjective.

Example 1.6. $M = T^*X$, where X is a smooth manifold. Define the Liouville 1-form η on T^*X as follows. Let $\pi\colon T^*X \to X$ be the projection map. Then given $v \in T_{(x,p)}(T^*X)$, we set $\eta(v) = (d\pi \cdot v, p)$. Thus if x_i are local coordinates on X and p_i are the linear coordinates in the fibers of T^*X with respect to the basis dx_i then $\eta = \sum p_i dx_i$.

Let $\omega = d\eta$. Then ω is a symplectic structure on M. In local coordinates, $\omega = \sum dp_i \wedge dx_i$.

Example 1.7. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $\Pi\colon \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ be the dual map to the Lie bracket. Then Π is a Poisson bivector on $\mathfrak{g}^*$ (whose coefficients are linear). This Poisson structure on $\mathfrak{g}^*$ is called the Lie Poisson structure.

Let $\mathcal{O}$ be an orbit of the coadjoint action in $\mathfrak{g}^*$. Then it is easy to check that the restriction of Π to $\mathcal{O}$ is a section of $\wedge^2 T\mathcal{O}$, which is nondegenerate. Thus $\mathcal{O}$ is a symplectic manifold. The symplectic structure on $\mathcal{O}$ is called the Kirillov–Kostant structure.

1.2 Moment maps

Let M be a Poisson manifold and G a Lie group acting on M by Poisson automorphisms. Let $\mathfrak{g}$ be the Lie algebra of G. Then we have a homomorphism of Lie algebras $\phi\colon \mathfrak{g} \to \mathrm{Vect}_\Pi(M)$.

Definition 1.8. A G-equivariant regular map $\mu\colon M \to \mathfrak{g}^*$ is said to be a moment map for the G-action on M if the pullback map $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ satisfies the equation $v(\mu^*(a)) = \phi(a)$.

It is easy to see that in this case μ^* is a homomorphism of Lie algebras, so μ is a Poisson map. Moreover, it is easy to show that if G is connected then the condition of G-equivariance in the above definition can be replaced by the condition that μ is a Poisson map.

A moment map does not always exist, and if it does, it is not always unique. However, if M is a simply connected symplectic manifold, then the homomorphism $\phi\colon \mathfrak{g} \to \mathrm{Vect}_\Pi(M)$ can be lifted to a homomorphism $\hat{\mathfrak{g}} \to C^\infty(M)$, where M is a 1-dimensional central extension of $\mathfrak{g}$. Thus there exists a moment map for the action on M of the simply connected Lie group $\widehat{G}$ corresponding to the Lie algebra $\hat{\mathfrak{g}}$. In particular, if in addition the action of G on M is transitive, then M is a coadjoint orbit of $\widehat{G}$.

We also see that if M is a connected symplectic manifold then any two moment maps $M \to \mathfrak{g}^*$ differ by shift by a character of $\mathfrak{g}$.

Exercise 1.9. Show that if $M = \mathbb{R}^2$ with symplectic form $dp \wedge dx$ and $G = \mathbb{R}^2$ acting by translations, then there is no moment map $M \to \mathfrak{g}^*$. What is $\widehat{G}$ in this case?

Exercise 1.10. Show that if M is simply connected and symplectic and G is compact then there is a moment map $M \to \mathfrak{g}^*$.

Exercise 1.11. Show that if M is symplectic then μ is a submersion near x (i.e., the differential $d\mu_x\colon T_xM \to \mathfrak{g}^*$ is surjective) if and only if the stabilizer G_x of G is a discrete subgroup of G (i.e., the action is locally free near x).

Example 1.12. Let $M = T^*X$, and let G act on X. Define $\mu\colon T^*X \to \mathfrak{g}^*$ by $\mu(x, p)(a) = p(\psi(a))$, $a \in \mathfrak{g}$, where $\psi\colon \mathfrak{g} \to \mathrm{Vect}(X)$ is the map defined by the action. Then μ is a moment map.

1.3 Hamiltonian reduction

Let M be a Poisson manifold with an action of a Lie group G preserving the Poisson structure, and with a moment map μ. Then the algebra of G-invariants $C^\infty(M)^G$ is a Poisson algebra.

Let J be the ideal in $C^\infty(M)$ generated by $\mu^*(a)$, $a \in \mathfrak{g}$. It is easy to see that J is invariant under Poisson bracket with $C^\infty(M)^G$. Therefore, the ideal J^G in $C^\infty(M)^G$ is a Poisson ideal, and hence the algebra $A := C^\infty(M)^G/J^G$ is a Poisson algebra.

The geometric meaning of the algebra A is as follows. Assume that the action of G on M is proper, i.e., for any two compact sets K_1 and K_2 the set of elements $g \in G$ such that $gK_1 \cap K_2 \neq \emptyset$ is compact. Assume also that the action of G is free. In this case, the quotient M/G is a manifold, and $C^\infty(M)^G = C^\infty(M/G)$. Moreover, as we mentioned in Exercise 1.11, the map μ is a submersion (so $\mu^{-1}(0)$ is a smooth submanifold of M), and the ideal J^G corresponds to the submanifold $M//G := \mu^{-1}(0)/G$ in M. Thus $A = C^\infty(M//G)$, and so $M//G$ is a Poisson manifold.

Definition 1.13. The manifold $M//G$ is called the Hamiltonian reduction of M with respect to G using the moment map μ.

Exercise 1.14. Show that in this setting, if M is symplectic, so is $M//G$.

This geometric setting can be generalized in various directions. First of all, for $M//G$ to be a manifold, it suffices to require that the action of G be free only near $\mu^{-1}(0)$. Second, one can consider a locally free action which is not necessarily free. In this case, $M//G$ is a Poisson orbifold.

Finally, we can consider a purely algebraic setting which will be most convenient for us: M is a scheme of finite type over $\mathbb{C}$ (for example, a variety), and G is an affine algebraic group. In this case, we do not need to assume that the action of G is locally free (which allows us to consider many more examples). Still, some requirements are needed to ensure the existence of quotients. For example, a sufficient condition that often applies is that M is an affine scheme and G is a reductive group. Then $M//G$ is an affine Poisson scheme (possibly non-reduced and singular even if M was smooth).

Example 1.15. Let G act properly and freely on a manifold X, and $M = T^*X$. Then $M//G$ (for the moment map as in Example 1.12) is isomorphic to $T^*(X/G)$.

On the other hand, the following example shows that when the action of G on an algebraic variety X is not free, the computation of the reduction $T^*X//G$ (as a scheme) may be rather difficult.

Example 1.16. Let $M = T^*\operatorname{Mat}_n(\mathbb{C})$, and $G = \mathrm{PGL}_n(\mathbb{C})$ (so $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$). Using the trace form we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and M with $\operatorname{Mat}_n(\mathbb{C}) \oplus \operatorname{Mat}_n(\mathbb{C})$. Then a moment map is given by the formula $\mu(X, Y) = [X, Y]$, for $X, Y \in \operatorname{Mat}_n(\mathbb{C})$. Thus $\mu^{-1}(0)$ is the *commuting scheme* $\operatorname{Comm}(n)$ defined by the equations $[X, Y] = 0$, and the quotient $M//G$ is the quotient $\operatorname{Comm}(n)/G$, whose ring of functions is $A = \mathbb{C}[\operatorname{Comm}(n)]^G$.

It is not known whether the commuting scheme is reduced (i.e. whether the corresponding ideal is a radical ideal); this is a well-known open problem. The underlying variety is irreducible (as was shown by Gerstenhaber [Ge1]), but very singular, and has a very complicated structure. However, we have the following result.

Theorem 1.17 (Gan, Ginzburg, [GG]). *$\operatorname{Comm}(n)/G$ is reduced, and isomorphic to $\mathbb{C}^{2n}/S_n$. Thus $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$. The Poisson bracket on this algebra is induced from the standard symplectic structure on $\mathbb{C}^{2n}$.*

Remark. The hard part of this theorem is to show that $\operatorname{Comm}(n)/G$ is reduced (i.e., A has no nonzero nilpotent elements).

Remark. Let $\mathfrak{g}$ be a simple complex Lie algebra, and G the corresponding group. The commuting scheme $\operatorname{Comm}(\mathfrak{g})$ is the subscheme of $\mathfrak{g} \oplus \mathfrak{g}$ defined by the equation $[X, Y] = 0$. Similarly to the above discussion, $\operatorname{Comm}(\mathfrak{g})/G = T^*\mathfrak{g}//G$. It is conjectured that $\operatorname{Comm}(\mathfrak{g})$ and in particular $\operatorname{Comm}(\mathfrak{g})/G$ is a reduced scheme; the latter is known for $\mathfrak{g} = \mathfrak{sl}_n$ thanks to the Gan–Ginzburg theorem. It is also known that the underlying variety $\overline{\mathrm{Comm}(\mathfrak{g})}$ is irreducible (as was shown by Richardson), and $\overline{\mathrm{Comm}(\mathfrak{g})}/G = (\mathfrak{h} \oplus \mathfrak{h})/W$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and W is the Weyl group of $\mathfrak{g}$ (as was shown by Joseph [J]).

1.4 Hamiltonian reduction along an orbit

Hamiltonian reduction along an orbit is a generalization of the usual Hamiltonian reduction. For simplicity let us describe it in the situation when M is an affine algebraic variety and G a reductive group. Let $\mathcal{O}$ be a closed coadjoint orbit of G, $I_{\mathcal{O}}$ be the ideal in $S\mathfrak{g}$ corresponding to $\mathcal{O}$, and let $J_{\mathcal{O}}$ be the ideal in $\mathbb{C}[M]$ generated by $\mu^*(I_{\mathcal{O}})$. Then $J_{\mathcal{O}}^G$ is a Poisson ideal in $\mathbb{C}[M]^G$, and $A = \mathbb{C}[M]^G/J_{\mathcal{O}}^G$ is a Poisson algebra.

Geometrically, $\mathrm{Spec}(A) = \mu^{-1}(\mathcal{O})/G$ (categorical quotient). It can also be written as $\mu^{-1}(z)/G_z$, where $z \in \mathcal{O}$ and G_z is the stabilizer of z in G.

Definition 1.18. The scheme $\mu^{-1}(\mathcal{O})/G$ is called the Hamiltonian reduction of M with respect to G along $\mathcal{O}$. We will denote it by $R(M, G, \mathcal{O})$.

Exercise 1.19. Show that if the action of G on $\mu^{-1}(\mathcal{O})$ is free and M is a symplectic variety, then $R(M, G, \mathcal{O})$ is a symplectic variety of dimension $\dim(M) - 2\dim(G) + \dim(\mathcal{O})$.

Exercise 1.20 (The Duflo–Vergne theorem, [DV]). Let $z \in \mathfrak{g}^*$ be a generic element. Show that the connected component of the identity of the group G_z is commutative.

Hint. The orbit $\mathcal{O}$ of z is described locally near z by equations $f_1 = \cdots = f_m = 0$, where the f_i are Casimirs of the Poisson structure (i.e. functions that Poisson commute with any function); the Lie algebra $\mathrm{Lie}(G_z)$ has basis $df_i(z)$, $i = 1, \ldots, m$.

In a similar way, one can define Hamiltonian reduction along any Zariski closed G-invariant subset of $\mathfrak{g}^*$, for example the closure of a non-closed coadjoint orbit.

1.5 Calogero–Moser space

Let M and G be as in Example 1.16, and $\mathcal{O}$ be the orbit of the matrix $\mathrm{diag}(-1, -1, \ldots, -1, n-1)$, i.e. the set of traceless matrices T such that $T + 1$ has rank 1.

Definition 1.21 (Kazhdan, Kostant, Sternberg, [KKS]). The scheme

$$\mathcal{C}_n := R(M, G, \mathcal{O})$$

is called the Calogero–Moser space.