PROCEEDINGS OF SYMPOSIA IN APPLIED MATHEMATICS

Volume VII

PROCEEDINGS OF
SYMPOSIA IN APPLIED MATHEMATICS

VOLUME VII

APPLIED PROBABILITY

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK TORONTO LONDON

1957

FOR THE AMERICAN MATHEMATICAL SOCIETY

80 WATERMAN STREET, PROVIDENCE, RHODE ISLAND

PROCEEDINGS OF THE SEVENTH SYMPOSIUM IN APPLIED MATHEMATICS OF THE AMERICAN MATHEMATICAL SOCIETY

Held at the Polytechnic Institute of Brooklyn
April 14–15, 1955

COSPONSORED BY

THE OFFICE OF ORDNANCE RESEARCH

L. A. MacColl

EDITOR

EDITORIAL COMMITTEE

R. V. Churchill
A. E. Heins
F. J. Murray

Prepared by the American Mathematical Society under Contract No. DA-19-020-ORD-3538 with the Ordnance Corps, U.S. Army.

Library of Congress Catalog Card Number 50-1183

EDITOR'S PREFACE

The Seventh Symposium in Applied Mathematics, sponsored by the American Mathematical Society and the Office of Ordnance Research, and devoted to *Mathematical Probability and Its Applications*, was held at the Polytechnic Institute of Brooklyn on April 14 and 15, 1955. This volume contains the papers (one in abstract form) which were presented at the Symposium.

Prolonged consideration by the members of the Program Committee, under the chairmanship of Dr. H. W. Bode, resulted in the decision that the Symposium should be concerned with three principal themes, *viz.*, *The Theory of Diffusion*, *The Theory of Turbulence*, and *Probability in Classical and Modern Physics*. However, it was the intention of the Committee that these terms should be interpreted broadly and that the speakers should avail themselves of considerable freedom in determining the actual contents of their papers. In particular, it was understood that the term "theory of diffusion" was to be interpreted so as to cover a wide variety of relations between probability and differential equations.

The three themes were dealt with in the order in which they have been mentioned, and the papers appear here in the order in which they were given.

Many individuals have participated, directly and indirectly, in the work of preparing this volume. The editor wishes to express here his sincere thanks to all of these collaborators. The advice and encouragement given by Professor R. V. Churchill, Chairman of the Editorial Committee for the Proceedings of Symposia in Applied Mathematics, has been particularly helpful. All who participated in the Symposium are indebted to the McGraw-Hill Book Company, Inc., which, beginning with the Proceedings of the Symposium on Elasticity, has undertaken the task of bringing the Proceedings of these Symposia on Applied Mathematics to the scientific public in book form.

L. A. MacColl

Editor

CONTENTS

BROWNIAN MOTION DEPENDING ON n PARAMETERS: THE PARTICULAR CASE $n = 5$

BY

PAUL LEVY

INTRODUCTION

Let E_n be the Euclidean n-dimensional space, A and B points of E_n, and $X(A)$ the Brownian function, *i.e.*, the Gaussian random function (r.f.) defined up to an additive constant by the formula

$$X(B) - X(A) = \xi[r(A,B)]^{\frac{1}{2}}, \tag{1}$$

in which $r(A,B)$ is the distance between A and B, and ξ is a reduced Gaussian variable.

Let us now suppose that a set $\mathcal{E} \subset E_n$ is given and that the values of $X(A)$ are known at every $A \in \mathcal{E}$. At the other points $X(A)$ has a conditional distribution, which we shall write in the canonical form $m + \sigma\xi$, where $m = E\{X(A)|\mathcal{E}\}$ and $\sigma = \sigma\{X(A)|\mathcal{E}\}$ are the conditional expectation and the conditional standard deviation. If we obtain new information, i.e., if $\mathcal{E}$ increases, σ is nondecreasing.

Now let Ω_t be the surface of a sphere with center O and radius t, and let $\bar{M}(t)$ be the mean value of $X(A)$ on Ω_t. Although $\bar{M}(t)$ is not a well-defined r.f., $M(t) = \bar{M}(t) - X(O)$ is a well-defined Gaussian r.f. Thus, to redefine $M(t)$, it is sufficient to know its covariance $\Gamma_n(t_1,t_2)$.

Let $\mathcal{E}_t$ denote the entire region outside of Ω_t, that is, $r(O,A) \geqq t$, and let

$$\sigma\{X(O)|\mathcal{E}_t\} = c_n t^{\frac{1}{2}}, \qquad \sigma\{X(O)|\Omega_t\} = c'_n t^{\frac{1}{2}}. \tag{2}$$

Since σ is a nonincreasing function of $\mathcal{E}$, $c_n t^{\frac{1}{2}}$ is the minimum of $\sigma\{X(O)|\mathcal{E}\}$ if $\mathcal{E} \subset \Omega_t$. Since E_n is a section of E_{n+1}, one obviously has

$$c_n \leqq c'_n, \qquad c_n \geqq c_{n+1}, \qquad c'_n \geqq c'_{n+1}. \tag{3}$$

Then, in the limiting case $n = \infty$ (Hilbert space), c_n and c'_n have limits c_ω and c'_ω. These limits are

$$c_\omega = 0, \qquad c'^2_\omega = 1 - 2^{-\frac{1}{2}}. \tag{4}$$

It is easy to find the value of c'_ω but not easy to obtain the values of c_n and to prove that $\lim c_n = 0$. Since $m\{X(O)|\mathcal{E}_t\}$ is obviously a linear function of the values of $\bar{M}(u)$ in $(0,t)$, we were led to consider the continuation to the left of $\bar{M}(u)$. Thus the first problem is to define the covariance $\Gamma_n(t_1,t_2)$.

If n is even, the covariance is an elliptic integral, and the study of $M(t)$ appears to be very difficult. If n is odd ($n = 2p + 1$), then Γ_n is a rational function of $t = \min(t_1,t_2)$ and $t' = \max(t_1,t_2)$. Thus one can obtain an explicit expression for $M(t)$ and solve all problems concerning the continuation of $\bar{M}(t)$ to the left and to the right.

But the subject is wide, and this is only the first of several papers.† The second will be presented at the next Berkeley Symposium. In this first paper, the particular case $n = 5$ will be completely described. The case $n = 3$ is too simple to give satisfactory insight into the general theory. The value $n = 5$ presents for the first time a very curious circumstance: $M(t)$ satisfies two different stochastic differential equations; the first determines the continuation to the right, and the second the continuation to the left. These two equations are related to two analytic expressions for $M(t)$.

The author was very surprised to notice these facts, and thinks it worthwhile to begin with a preliminary chapter in which a quite analogous circumstance appears in a very simple case. After this chapter, it will be easier to follow the solution of our problem in the case $n = 5$. Brief indications on the general case $n = 2p + 1$ will be given in Sec. 2.3 and in the footnotes.

1. A Simple Class of Gaussian Random Functions

1.1. The Greek letters ξ and η, with or without subscripts, will always denote reduced Gaussian variables. The well-known Brownian (or Bachelier-Wiener) function will be denoted by

$$X(t) = \int_0^t \xi_u (du)^{\frac{1}{2}}, \qquad (t > 0). \tag{5}$$

This formula has the same meaning as

$$\delta X(t) = \xi_t (dt)^{\frac{1}{2}}, \tag{6}$$

where ξ_t depends on t and dt and the Cauchy condition $X(O) = 0$. Then, if $X_u = \lambda \xi_u$, with λ a constant, we may write

$$\int_0^t f(u) X_u (du)^{\frac{1}{2}} = \lambda \int_0^t f(u)\, dX(u) = \lambda f(t) X(t) - \lambda \int_0^t X(u)\, df(u).$$

Now let

$$\Psi(t) = \int_0^t (tX_u + uY_u)(du)^{\frac{1}{2}}, \qquad (t > 0), \tag{7}$$

where X_u and Y_u are joint Gaussian variables, with $E(X_u) = E(Y_u) = 0$, and

$$E(X_u^2) = a, \qquad E(X_u Y_u) = b, \qquad E(Y_u^2) = c, \tag{8}$$

where a, b, c are independent of u, and necessarily

$$a \geqq 0, \qquad c \geqq 0, \qquad b^2 \leqq ac; \tag{9}$$

moreover, unless $\Psi(t)$ is identically zero,

$$a + c > 0. \tag{9a}$$

† The most important results, without proofs, have already been given in C. R. Acad. Sci. Paris vol. 239 (1954) pp. 1181, 1584; vol. 240 (1955) pp. 1043, 1308. This is the first part of the detailed statement.

If $b^2 = ac$, we may write $\Psi(t)$ in the simpler form

(10) $$\Psi(t) = \int_0^t (\lambda t + \mu u)\xi_u(du)^{\frac{1}{2}}, \qquad (\lambda^2 = a, \lambda\mu = b, \mu^2 = c).$$

Obviously, if either ξ_u is replaced by $-\xi_u$ or λ and μ are replaced by $-\lambda$ and $-\mu$, then we have the same random function.† Then we shall assume

(11) $$s = \lambda + \mu \geqq 0,$$

without loss of generality. In the case (10) we have

$$\delta\Psi(t) = \lambda\, dt \int_0^t \xi_u(du)^{\frac{1}{2}} + st\xi_t(dt)^{\frac{1}{2}}, \qquad (dt > 0),$$

and, if we use (5), then

(12) $$\delta\Psi(t) = \lambda X(t)\, dt + st\, \delta X(t)$$

even with $dt < 0$. This formula and the Cauchy condition $\Psi(0) = 0$ are equivalent to (10).

If $s = 0$, then $\Psi(t)$ is differentiable, with derivative $\lambda X(t)$. In this case, $\Psi(t)$, is a Markovian function of order 2, *i.e.*, the two joint functions $\Psi(t)$ and $\Psi'(t)$ form a Markovian system. In the general case, $\Psi(t)$ and $X(t)$ form a Markovian function. Also $\Psi(t)$ is a Markovian function in the trivial cases $\lambda = 0$ and $\mu = 0$ and also if $k = \lambda/(\lambda + \mu) = -1$ or -2. [This will result from formula (28).]

1.2. A Gaussian function is defined if its covariance is known. If

$$t = \min(t_1,t_2), \qquad t' = \max(t_1,t_2),$$

then the covariance of $\Psi(t)$ can be written

$$\gamma(t_1,t_2) = \int_0^t E[(tX_u + uY_u)(t'X_u + uY_u)]\, du$$
$$= t' \int_0^t (at + bu)\, du + \int_0^t (btu + cu^2)\, du.$$

Finally, setting

(13) $$\alpha = a + \frac{b}{2}, \qquad \beta = \frac{b}{2} + \frac{c}{3},$$

we obtain

(14) $$\gamma(t_1,t_2) = \alpha t^2 t' + \beta t^3, \quad [t = \min(t_1,t_2),\ t' = \max(t_1,t_2)].$$

In the case (10), the values of α and β are

(15) $$\alpha = \lambda^2 + \frac{\lambda\mu}{2}, \qquad \beta = \frac{\lambda\mu}{2} + \frac{\mu^2}{3}.$$

† We consider a random function (r.f.) as well defined if we know the set of all possible functions and the probability distribution in that set. If one of these functions is considered apart from the others, we shall say "an individual function is known (or given)." Thus, if ξ_u is replaced by $-\xi_u$, we change the individual function but the r.f. is the same.

One might have thought the $\Psi(t)$ form a family of r.f. depending on three parameters, a, b, c. We now see that we have only two parameters, α and β.

1.3. THEOREM 1. 1°. *The function $\gamma(t_1,t_2)$ in* (14) *is a covariance if, and only if,*

$$3\alpha + \beta \geqq 0, \qquad 3\beta + \alpha \geqq 0. \tag{16}$$

2°. *In this case $\gamma(t_1,t_2)$ is the covariance of a function $\Psi(t)$, which can, in general, be represented in the form* (7) *depending on one parameter and also represented in two distinct ways in the form* (10). *The only exception is the case*

$$(3\alpha + \beta)(3\beta + \alpha) = 0, \tag{17}$$

where $\Psi(t)$ has only one representation, which may be written in the form (10).

Now, to exclude the case $\Psi(t) \equiv 0$, we may add the condition

$$\alpha + \beta = \frac{\gamma(t,t)}{t^3} = E\left[\frac{\Psi^2(t)}{t^3}\right] > 0. \tag{18}$$

Then at least one of the numbers $3\alpha + \beta$ and $3\beta + \alpha$ [the sum of which is $4(\alpha + \beta)$] is positive.

Proof. 1°. If $\gamma(t_1,t_2)$ is the covariance of a r.f. $\Phi(t)$, $e^{-3u}\Phi(e^{2u})$ has the covariance

$$\alpha e^{-|u|} + \beta e^{-3|u|}, \qquad (u = u_1 - u_2); \tag{19}$$

and, conversely, if this function is a covariance, $\gamma(t_1,t_2)$ is a covariance. Now, by a theorem of A. Khintchine, an even function of $u_1 - u_2$ that has a continuous Fourier transform is a covariance (in fact a stationary covariance) if, and only if, the Fourier transform

$$\frac{\alpha}{1 + u^2} + \frac{3\beta}{9 + u^2} = \frac{3(3\alpha + \beta) + (\alpha + 3\beta)u^2}{(1 + u^2)(9 + u^2)} \tag{20}$$

is never negative. Thus the conditions (16) are necessary and sufficient. Q.E.D.

2°. We have to compute a, b, c from (9) and (13). Using (13) to eliminate b and c, the last equation (9) is

$$b^2 - ac = a^2 - (5\alpha + 3\beta)a + 4\alpha^2 \leqq 0. \tag{21}$$

We deduce from (16) that

$$D = (5\alpha + 3\beta)^2 - 16\alpha^2 = 3(3\alpha + \beta)(\alpha + 3\beta) \geqq 0. \tag{22}$$

Therefore, the equation

$$x^2 - (5\alpha + 3\beta)x + 4\alpha^2 = 0 \tag{23}$$

has real roots, a_1 and $a_2 \geqq a_1$, with $b^2 \leqq ac$ if, and only if, $a \in [a_1,a_2]$. From

$$a_1a_2 = 4\alpha^2 \geqq 0, \qquad a_1 + a_2 = \tfrac{1}{2}(\alpha + 3\beta) + \tfrac{3}{2}(3\alpha + \beta) \geqq 0 \tag{24}$$

it follows that $a_2 \geqq a_1 \geqq 0$. If a is chosen in $[a_1, a_2]$, and if b and c are computed from (13), two possibilities arise: either $a = a_1 = 0$, and then $\alpha = 0$ and $c = 3\beta \geqq 0$, or else $a > 0$, and then $c \geqq 0$ is deduced from $ac \geqq b^2$. In both cases, the conditions (9) and (16) are fulfilled.

It is obvious that we get the form (10) if, and only if, $b^2 = ac$, that is, $a = a_1$ or a_2. We have one solution if $D = 0$ and two if $D > 0$. Q.E.D.

1.4. We shall now consider only the form (10) of $\Psi(t)$. We shall write the letters λ, μ, Ψ, or X with the subscript i ($i = 1$ or 2) if we wish to specify that $a = \lambda^2$ has the value a_i. From

(25) $$s^2 = (\lambda + \mu)^2 = \alpha + 3\beta,$$

and from (11), we deduce that s has only one value (in the general case, we have $a + 2b + c = \alpha + 3\beta$). We introduce k, K, and s by

(26) $$\lambda = ks, \qquad \mu = (1 - k)s, \qquad \beta = K\alpha.$$

We find from (15)

$$K = \frac{3\lambda\mu + 2\mu^2}{6\lambda^2 + 3\lambda\mu} = \frac{3k(1-k) + 2(1-k)^2}{6k^2 + 3k(1-k)} = \frac{2 - k - k^2}{3(k + k^2)},$$

that is,

(27) $$k^2 + k = \frac{2}{3K + 1}.$$

We have real values k_1 and k_2 of k if

$$1 + \frac{8}{3K + 1} = \frac{3(K + 3)}{3K + 1} \geqq 0.$$

This condition is another form of (22), and k_1 and k_2 are linked by the involutive relation

(28) $$k_1 + k_2 = -1.$$

The double points $-\frac{1}{2}$ and ∞ of this involution correspond respectively to the values -3 and $-\frac{1}{3}$ of K (if $k = \infty$, $s = \lambda + \mu = 0$; if $k = -\frac{1}{2}$, $3\lambda + \mu = 0$).

Since $a = \lambda^2 = k^2 s^2$ and $a_2 \geqq a_1$, and taking into account (28), we have

$$k_2^2 - k_1^2 = k_1 - k_2 \geqq 0,$$

and

(29) $$k_1 \geqq -\tfrac{1}{2} \geqq k_2.$$

1.5. Consider now the equation (12). We deduce

(30) $$E\{[\delta\Psi(t)]^2\} = s^2 t^2 |dt| + o(dt), \qquad (dt \to 0)$$

and have a new proof of the following fact: if the r.f. $\Psi(t)$ is defined, s^2 has only one possible value. The same conclusion holds if an individual $\Psi(t)$ is given in a small interval (t_1, t_2). This can be deduced easily from well-known almost sure (a.s.) properties of $X(t)$.† Since λ has two possible values (unless $K = -3$ or $-\frac{1}{3}$), there is no hope of deducing this parameter from an individual $\Psi(t)$.

Formula (12) leads us to another important conclusion when we write it successively with the subscripts 1 and 2 and suppose $\Psi_1(t) = \Psi_2(t)$. Then

$$st\ \delta[X_2(t) - X_1(t)] = [\lambda_1 X_1(t) - \lambda_2 X_2(t)]\,dt + o(dt). \tag{31}$$

Thus the difference $X_2(t) - X_1(t)$ is differentiable (even if $s = 0$; in this case it is zero).

1.6. If an individual $X(u)$ is given in $(0,T)$, all the ξ_u are known in $(0,T)$, and the corresponding $\Psi(t)$ is given by (10) or (12). The converse problem is very important. The solution is given by the following theorem.

THEOREM 2. *Suppose an individual $\Psi(t)$ is given in $[0,T]$. 1°. If*

$$k = k_1 > k_2,$$

and $t \in (0,T)$, then only one $X(t)$ corresponds to the given $\Psi(t)$. 2°. The same conclusion holds if $k_1 = k_2$. 3°. If $k = k_2 < k_1$, $X(t)$ is not known but depends on a Gaussian variable c, with positive standard deviation $\sigma(c)$.

Proof. In all three cases, (12) may be written in the form

$$s\ \delta[t^k X(t)] = t^{k-1}\ \delta\Psi(t). \tag{32}$$

Thus, if $s \neq 0$, and if $X^*(t)$ is a particular solution of this equation, the general solution is

$$X(t) = X^*(t) + ct^{-k}, \tag{33}$$

and c is obviously either a known number or a Gaussian random variable with positive standard deviation $\sigma(c)$. As the given values of $\Psi(t)$ and the unknown values of $X(t)$ are joint Gaussian variables, there exists no other possibility.

1°. If $k = k_1 > k_2$, then $k > -\frac{1}{2}$, and

$$t^k X(t) \xrightarrow{\text{a.s.}} 0, \qquad (t > 0). \tag{34}$$

This is a Cauchy condition, from which we deduce

$$st^k X(t) \doteq \int_0^t u^{k-1}\,d\Psi(u) = t^{k-1}\Psi(t) - (k-1)\int_0^t u^{k-2}\Psi(u)\,du. \tag{35}$$

Since here $s \neq 0$, the first statement is proved.

2°. $k_1 = k_2$ implies $s = 0$ or $k = -\frac{1}{2}$. If $s = 0$, we have already deduced from (12) that $\lambda X(t)$ is the derivative of $\Psi(t)$. If $k = -\frac{1}{2}$, the above Cauchy condition holds in a generalized sense. One has

$$U^{-1}\int_0^U e^u X(e^{-2u})\,du \xrightarrow{\text{a.s.}} 0, \qquad (U \to +\infty). \tag{36}$$

This condition yields c.

† For instance, if $n \to \infty$, $2^n[X(t + 2^{-n}) - X(t)]^2$ tends a.s. to 1 in the Cesaro sense. Then $2^n[\Psi(t + 2^{-n}) - \Psi(t)]^2$ tends a.s. to s^2t^2 in the same sense.

3°. The value of $\sigma(c)$ will be found in Sec. 1.7 [see formula (44)]. It is positive, and this will complete the proof of the present theorem.

Another proof may be deduced from a theorem which may be briefly stated as follows. *If a function $f(t)$, given in $(0,T)$, is a possible function $X(t)$, then $f(t) + ct^{\alpha}$ $(c \neq 0)$ is also a possible function $X(t)$ if and only if $\alpha > \frac{1}{2}$.* Thus, if $k < -\frac{1}{2}$, that is, $k = k_2 < k_1$, we have no way to choose one of the functions $X(t)$ in (33) and say that it is the correct one.

Corollary 1. 1°. If $k = k_1$ (even if $k_1 = k_2$), we have exactly the same information, whether an individual $\Psi(t)$ is given in $[0,T]$ or the corresponding $X(t)$ is given in this interval.

2°. If $k = k_2 < k_1$, we have more information when $X(t)$ is given in $[0,T]$ than when $\Psi(t)$ is given in this interval.

1.7. The continuation of $\Psi(t)$ to the right. 1°. As a preliminary problem, suppose an individual $X(u)$ given in $(0,t)$. The problem is to write $\Psi(t')$, $(t' > t)$, in the canonical form $m + \sigma\xi$, m being the conditional expectation and σ the standard deviation. Since in

$$\Psi(t') = \int_0^t (\lambda t' + \mu u)\xi_u(du)^{\frac{1}{2}} + \int_t^{t'} (\lambda t' + \mu u)\xi_u(du)^{\frac{1}{2}}$$

the ξ_u in the first integral are known $[\xi_u(du)^{\frac{1}{2}} = \delta X(u)]$ and in the second integral they are independent of the given individual $X(u)$, we have

$$m = \int_0^t (\lambda t' + \mu u)\, dX(u), \tag{37}$$

$$\sigma\xi = \int_t^{t'} (\lambda t' + \mu u)\xi_u(du)^{\frac{1}{2}}. \tag{38}$$

It follows from (38) that

$$\sigma^2 = \int_t^{t'} (\lambda t' + \mu u)^2\, du = (t' - t)\left[\lambda^2 t'^2 + \lambda\mu t'(t' + t) + \frac{\mu^2}{3}(t'^2 + tt' + t^2)\right] \tag{39}$$

As expected, if $t' - t = dt$, then $m - \Psi(t) + \sigma\xi$ has the form (12).

2°. Suppose now an individual $\Psi(u)$ given in $(0,t)$. If $\Psi(u)$ is given in the form $\Psi_1(u)$, we have exactly the same information as if $X_1(u)$ were given, and the preceding conclusion holds: $m + \sigma\xi$ is again the canonical form of $\Psi(t')$, $(t' > t)$, and m may be written

$$m = \Psi(t) + \lambda(t' - t)X(t). \tag{40}$$

Then, taking into account (35), we have

$$\Psi(t') - \Psi(t) = k\frac{t' - t}{t}\int_0^t \left(\frac{u}{t}\right)^{k-1} d\Psi(u) + \sigma\xi, \tag{41}$$

where σ is given by (39).

3°. If now $k_1 > k_2$ and we use the form $\Psi_2(t)$ of $\Psi(t)$, then $X(t) = X_2(t)$. Since this function is not known, the number m defined by (37) is not the conditional expectation of $\Psi(t')$. However, m and $\sigma\xi$ are still independent, (40) holds, and we may suppose $T = t$ in (33). Thus we have

$$\sigma^2 = \lambda_2^2\sigma^2(c)t^{-2k_2}(t' - t)^2 + \sigma_2^2, \tag{42}$$

where $\sigma^2 = \sigma_1^2$ and σ_2^2 are computed from (39), written with the subscripts 1 and 2. From $\lambda = ks$ and $\mu = (1 - k)s$, we deduce

$$\sigma^2 = (t' - t)s^2\left[\frac{1 + k + k^2}{3}t'^2 + \frac{1 + k - 2k^2}{3}tt' + (1 - 2k + k^2)t^2\right]. \tag{43}$$

Also, from $k_1 + k_2 = -1$, $k_1^2 - k_2^2 = -(k_1 - k_2)$, we deduce

$$\sigma_1^2 - \sigma_2^2 = (k_1 - k_2)s^2t(t' - t)^2.$$

Using (42), we have finally

$$\sigma^2(c) = (k_1 - k_2)k_2^{-2}t^{k_2-k_1} > 0, \qquad (k_2 - k_1 = 1 + 2k_2 < 0). \tag{44}$$

Thus the last statement of Theorem 2 is proved. Moreover, if $u < t$, then the unknown part of $X_2(u)$ is $\sigma(c)\xi u^{-k_2}$, and its standard deviation, $\sigma(c)u^{-k_2}$, is a decreasing function of t, as foreseen, since, as t increases, we have more information.

1.8. The continuation of $\Psi(t)$ to the left. 1°. The formulas (32) and (33) hold if an individual $\Psi(t)$ is given in (T, ∞). But here, as $t \to \infty$, then $t^kX(t) \xrightarrow{\text{a.s.}} 0$ if, and only if, $k = k_2 < -\frac{1}{2}$ (instead of $k = k_1 > -\frac{1}{2}$). Thus we have a theorem which is quite like Theorem 2, but the subscripts 1 and 2 are interchanged. If $k = k_2$ (even in the limiting cases $k = -\frac{1}{2}$ or ∞), $X(t) = X_2(t)$ is known in (T, ∞), and if $k < -\frac{1}{2}$, it is given by

$$st^kX_2(t) = \int_\infty^t u^{k-1}\, d\Psi(u). \tag{45}$$

On the contrary, if $k = k_1 > -\frac{1}{2}$, $X(t) = X_1(t)$ is a r.f., and may be written in the form (33), with c again a Gaussian variable.

2°. Since $X_2(t)$ depends only on $d\Psi(u)$, the preceding results hold if the difference $\Psi(t) - \Psi(T)$, instead of $\Psi(t)$, is known in (T, ∞). Then $X_2(t)$ is known in (T, ∞), and not only are the ξ_u with subscripts $u > T$ known in $X_2(t)$, but also the others are linked by

$$J_0 = \int_0^T \xi_u(du)^{\frac{1}{2}} = X_2(T).$$

The continuation problem is not so simple as in Sec. 1.7. We have to write $\Psi(t)$, $(t \leq T)$ in the canonical form $m + \sigma\xi$. From the general theory of joint Gaussian variables, we know that $m = ctJ_0$ and that c is given by the condition that σ^2 should be minimum. One has

$$\sigma^2 = E\{[\Psi(t) - cJ_0]^2\} = \int_0^t [(\lambda - c)t + \mu u]^2\, du + c^2t^2 \int_t^T du$$
$$= (\lambda - c)^2t^3 + (\lambda - c)\mu t^3 + \frac{\mu^2}{3} t^3 + c^2t^2(T - t)$$
$$= c^2t^2T - (2\lambda + \mu)ct^3 + \left(\lambda^2 + \lambda\mu + \frac{\mu^2}{3}\right) t^3.$$

Thus the minimum is obtained with

(46) $$c = \left(\lambda + \frac{\mu}{2}\right)\frac{t}{T}, \qquad m = \left(\lambda + \frac{\mu}{2}\right)\frac{t^2}{T} X_2(T),$$

and its value is

(47) $$\sigma^2 = \left(\lambda^2 + \lambda\mu + \frac{\mu^2}{3}\right) t^3 - \left(\lambda + \frac{\mu}{2}\right)^2 \frac{t^4}{T}$$
$$= \frac{\mu^2}{12} t^3 + \left(\lambda + \frac{\mu}{2}\right)^2 t^3\left(1 - \frac{t}{T}\right).$$

If $t = 0$, then $m = \sigma = 0$. If $t = T$, then the canonical form for $\Psi(T)$ is

(48) $$\Psi(T) = \left(\lambda + \frac{\mu}{2}\right) TX_2(T) + \sigma\xi, \qquad \left(\sigma^2 = \frac{\mu^2}{12} T^3\right),$$

and, if $\lambda = 0$, $\mu = 1$,

(49) $$J_1 = \int_0^T u\xi_u(du)^{\frac{1}{2}} = \frac{T}{2} X_2(T) + \frac{T}{2}\left(\frac{T}{3}\right)^{\frac{1}{2}} \xi,$$

from which one easily returns to (48).

All these formulas hold, if $X(T)$ is known, for λ_i, μ_i, X_i, either with $i = 1$ or with $i = 2$. However, we have supposed only that an individual $\Psi(t) - \Psi(T)$ is given in (T, ∞) and that $X_2(t)$ is given by (45) but not $X_1(t)$ unless $\lambda_1 = \lambda_2$. Thus these formulas give the canonical forms of $\Psi(t)$ and $\Psi(T)$ if $i = 2$ but not if $i = 1$, unless $k_1 = k_2$.

3°. If $k = k_1 > -\frac{1}{2}$, we deduce from (33) and (48)

$$\Psi(T) = \frac{k_1 + 1}{2} sTX^*(T) + \frac{k_1 + 1}{2} csT^{1-k_1} + \sigma_1\xi,$$

and, since we have necessarily the same $\sigma = \sigma_2$ as in (47), thus

$$(k_1 + 1)^2 \frac{s^2}{4} \sigma^2(c) = \sigma_2^2 - \sigma_1^2 = (k_1 - k_2) \frac{s^2T^2}{4},$$

and finally

(50) $$\sigma^2(c) = \frac{(k_1 - k_2)}{(k_1 + 1)^2} T^2 = \frac{k_1 - k_2}{k_2^2} T^2,$$

(51) $$\sigma^2[X_1(t)] = \frac{k_1 - k_2}{k_2^2} t^{-2k_1}T^2, \qquad (t \geqq T).$$

As foreseen, this value increases with T.

4°. The same method may be used for other problems. If, for instance, only $X(t) - X(T)$ is given in (T, ∞), we have no information on the ξ_u with subscripts $u < t$, and the problem becomes quite trivial. On the contrary, if $\mu \neq 0$, and if $\Psi(t)$ is given in (T, ∞), then $X_2(t)$, $(t \geq T)$, J_0 and J_1 are known, and the conditional expectation of $\Psi(t)$, $(t < T)$ has the form $c_0J_0 + c_1J_1$. The calculation is more complicated than in 2° but not difficult. If $\mu = 0$. then $\Psi(t) = \lambda t X(t)$, and this problem is trivial.

1.9. Application of the Fourier-Wiener series. It is known that the complex Brownian function

$$Z(t) = \frac{X(t) + iY(t)}{2^{\frac{1}{2}}} = \int_0^t \zeta_u (du)^{\frac{1}{2}}$$

may be defined in $(0,2\pi)$ by the Paley-Wiener formula

$$Z(t) = (2\pi)^{-\frac{1}{2}} \left[\zeta t + \sum_{n \neq 0} n^{-1} \zeta_n (e^{int} - 1) \right], \tag{52}$$

where $\zeta = 2^{-\frac{1}{2}}(\xi + i\eta)$ and the $\zeta_n = 2^{-\frac{1}{2}}(\xi_n + i\eta_n)$ are independent. Integrating from 0 to t, we get

$$\int_0^t (t - u)\zeta_u (du)^{\frac{1}{2}} = (2\pi)^{-\frac{1}{2}} \left[\zeta \frac{t^2}{2} + \sum_{n \neq 0} \frac{\zeta_n}{in^2} (e^{int} - 1 - int) \right]. \tag{53}$$

Now, from

$$\int_0^t (\lambda t + \mu u)\zeta_u (du)^{\frac{1}{2}} = stZ(t) - \mu \int_0^t (t - u)\zeta_u (du)^{\frac{1}{2}}, \tag{54}$$

and from (52) and (53), we obtain the integral on the left side of (54) [the real part of which is $\Psi(t)$] in the form of a series.

1.10. Generalization. Instead of $\Psi(t)$, let us consider the r.f.

$$\int_0^t (t^h X_{0,u} + t^{h-1} u X_{1,u} + \cdots + u^h X_{h,u})(du)^{\frac{1}{2}} \tag{55}$$

and the particular case

$$\int_0^t (\lambda_0 t^h + \lambda_1 t^{h-1} u + \cdots + \lambda_h u^h)\xi_u (du)^{\frac{1}{2}}. \tag{56}$$

Although the integral (55), where the $X_{k,u}$ are defined by formulas like (8), introduces $(h + 1)(h + 2)/2$ parameters, only $h + 1$ of them are essential, and every r.f. of the form (55) has, in general, several distinct representations of the form (56). If the condition (11) is replaced by $\Sigma\lambda_\nu \geq 0$, one has at first sight 2^{h-1} distinct representations. But they may introduce complex numbers, and the problem is to know how many are real; perhaps never more than two.

These random functions are related to the function $M(t)$ in the space E_{2h+3} in the same way as the functions (7) and (10) are related to $M(t)$ in the case $n = 5$. We shall see in Chap. 2 that $M(t)$ has two distinct expressions, connected with two distinct stochastic differential equations (see Sec. 2.8). The same result probably holds for the more general r.f. defined by formula (56).

2. The Random Function $M(t)$

2.1. Definitions and general remarks. Let us consider, as in the Introduction, the Euclidean space E_n and the Brownian function $X(A)$, defined up to an additive constant by the formula

$$X(B) - X(A) = \xi[r(A,B)]^{\frac{1}{2}}, \tag{57}$$

where $r(A,B)$ is the distance between A and B.

We consider now a family of closed surfaces Ω_t, depending on a parameter t, and denote by $\bar{M}(t)$, $M(t)$, $U_t(B)$ and $\rho_n(t,t')$ the mean values of $X(A)$, $X(A) - X(O)$, $r(A,B)$, and $U_{t'}(A)$ as A describes Ω_t. Since obviously

$$2E\{[X(A) - X(O)][X(B) - X(O)]\} = r(O,A) + r(O,B) - r(A,B),$$

the covariance of $M(t)$ is

$$\Gamma_n(t,t') = E[M(t)M(t')] = \tfrac{1}{2}[U_t(O) + U_{t'}(O) - \rho_n(t,t')]. \tag{58}$$

General theorems on $\Gamma(t,t')$ are easy to deduce, especially if n is an odd number, from the relation of $U_t(B)$ with the Newtonian potential. In this paper we shall consider only the particular case in which Ω_t is a sphere with center O and radius t, and especially the case $n = 5$. Thus we need no general theory.

When Ω_t is the considered sphere, we have

$$\Gamma_n(t,t') = \tfrac{1}{2}[t + t' - \rho_n(t,t')], \tag{59}$$

and, setting

$$I_h = \int_0^{\pi/2} \sin^h \theta \, d\theta, \qquad r^2 = t^2 + t'^2 - 2tt' \cos \theta, \tag{60}$$

we have

$$\rho_n = \frac{1}{2I_{n-2}} \int_0^{\pi} r \sin^{n-2} \theta \, d\theta, \tag{61}$$

so that, if $n = 5$,

$$\rho_5 = \tfrac{3}{4} \int_0^{\pi} r \sin^3 \theta \, d\theta. \tag{62}$$

We see at once, taking $\cos \theta$ or r as parameter, that if n is an even number, ρ_n is an elliptic integral. It is only when n is an odd number that ρ_n and Γ_n are elementary functions. Then we can easily obtain explicit expressions for $M(t)$. As already stated, we shall consider here only the case $n = 5$, and write $\Gamma(t,t')$ instead of $\Gamma_5(t,t')$.

2.2. The covariance $\Gamma(t_1,t_2)$. In this paper t_1 and t_2 will denote two nonnegative numbers, and we shall set

$$t = \min(t_1,t_2), \qquad t' = \max(t_1,t_2).$$

Then $t' \geqq t$, and $\Gamma(t_1,t_2) = \Gamma(t,t')$.

From (62), choosing r as a new parameter and taking account of

$$r\,dr = tt'\sin\theta\,d\theta,$$

we deduce

$$\rho_5 = \frac{3}{16t^3t'^3}\int_{t'-t}^{t'+t}[-(t'^2-t^2)^2+2(t^2+t'^2)r^2-r^4]r^2\,dr$$
$$= \frac{3}{16t^3t'^3}\left[-(t'^2-t^2)^2\frac{r^3}{3}+2(t^2+t'^2)\frac{r^5}{5}-\frac{r^7}{7}\right]_{t'-t}^{t'+t},$$

and finally

(63) $$\rho_5 = t'+\frac{2t^2}{5t'}-\frac{t^4}{35t'^3}.\dagger$$

Then we deduce from (59)

(64) $$\Gamma(t_1,t_2) = \frac{t}{2}-\frac{t^2}{5t'}+\frac{t^4}{70t'^3}.$$

If we put $t = e^{2u}$, then $M(t)t^{-\frac{1}{2}}$ is a Gaussian stationary function, with covariance

$$\frac{\Gamma(t_1,t_2)}{(t_1t_2)^{\frac{1}{2}}} = \tfrac{1}{2}e^{-|u|}-\tfrac{1}{5}e^{-3|u|}+\tfrac{1}{70}e^{-7|u|}, \qquad (u = u_1-u_2).$$

This function and its first four derivatives are continuous, but the fifth derivative has a jump at the point $u = 0$.‡

2.3. The continuity theorem for $M(t)$. Consider the function

(65) $$cI^pX(t) = c\int_0^t\frac{(t-u)^p}{p!}\,dX(u) = c\int_0^t\frac{(t-u)^p}{p!}\,\xi_u(du)^{\frac{1}{2}},$$

the pth derivative of which is $cX(t)$. Its covariance is

$$\frac{c^2}{(p!)^2}\int_0^t(t-u)^p(t'-u)^p\,du,$$

and when the sign of t_2-t_1 changes, the difference between this function written for $t_2>t_1$ and the analytical continuation of the function written for

† If $n = 2p+1$, one has more generally

$$\Gamma(t_1,t_2) = \frac{t}{2}-c_1\frac{t^2}{t'}+c_2\frac{t^4}{t'^3}-\cdots+(-1)^pc_p\frac{t^{2p}}{t'^{2p-1}},$$

and it is easy to deduce $c_1, c_2, \cdots, c_p$ from a continuity theorem on $U_t(A)$.

‡ This is a particular case of the continuity theorem: if $n = 2p+1$, the difference

$$\Gamma_n(t_1,t_2)-(-1)^p\frac{|t_2-t_1|^n}{4nt_1^pt_2^p}, \qquad (t_1,\ t_2>0),$$

and all its derivatives of order $\leqq n$ are continuous.