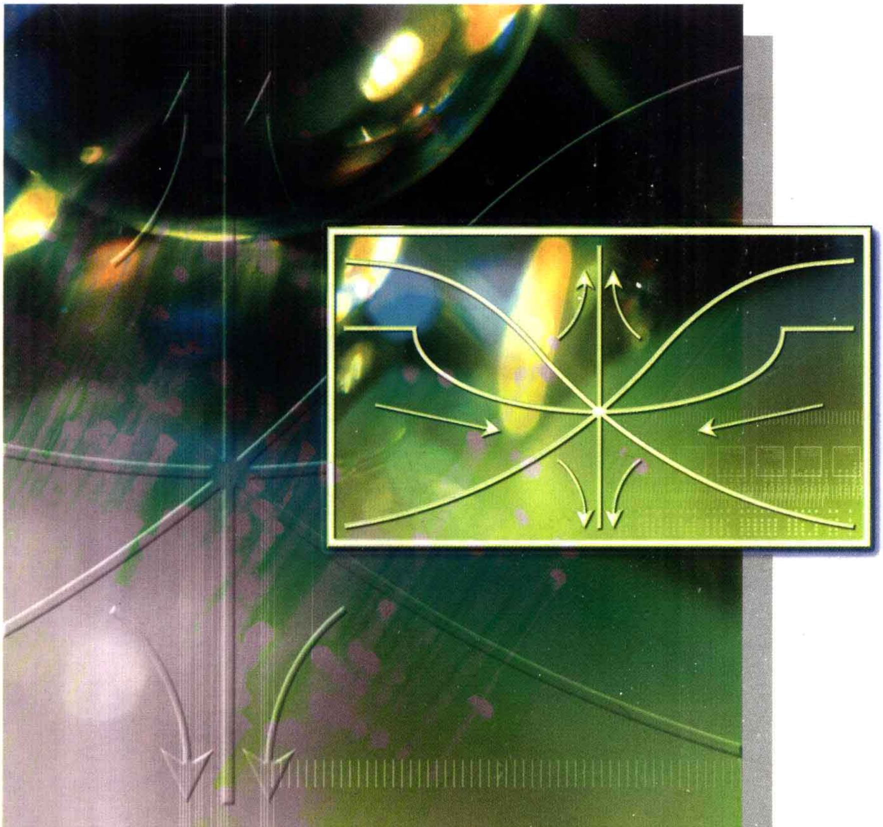
METHODS OF WAVE THEORY IN DISPERSIVE MEDIA

M V Kuzelev • A A Rukhadze



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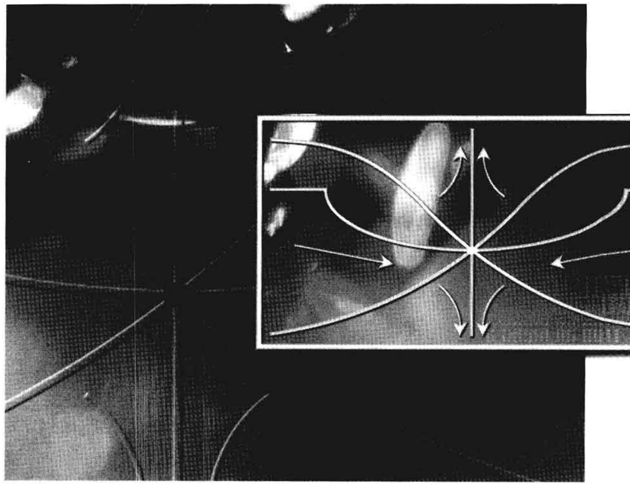

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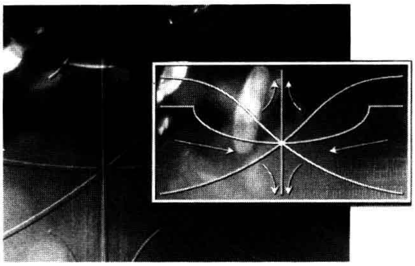

METHODS OF WAVE THEORY IN DISPERSIVE MEDIA

Synopsis

The monograph presents main analytic mathematical methods and general problems in the theory of linear waves in dispersive media and systems, including nonequilibrium ones. To show how the general theory can be applied in practice, a unified description is given of important physical systems that are traditionally studied in the mechanics of continuous media, electrodynamics, plasma physics, electronics, and physical kinetics. An analysis is made of the interaction of waves in coupled systems, the propagation and evolution of localized wave perturbations, and the emission of waves in dispersive media under the action of external sources moving in a prescribed manner. A general theory of instabilities of linear systems is presented in which the criteria for absolute and convective instabilities are formulated and compared, and Green's functions for some nonequilibrium media are calculated. Special attention is paid to problems in the theory of linear electromagnetic waves in plasmas and plasmalike media. The monograph also contains a number of original results of the present-day wave theory that have been published by now in scientific journals only.

The book is aimed at researchers and experts, as well as students and postgraduates, who specialize in such fields as the electrodynamics and mechanics of continuous media, physical electronics, and radiophysics.

Introduction

The monograph presents mathematical methods of description and general physical results in the theory of linear waves in dispersive media and systems, including nonequilibrium ones. In essence, it gives formulations and solutions of problems for nth order linear partial differential equations and also physical interpretations of the solutions and their practical applications. Since the literature (manuals, monographs, reviews, etc.) on the theory of linear waves is now so extensive that it seems to be exhaustive, the question naturally arises of whether it is expedient to publish books like the one you are reading. We think, however, that books of this kind are still needed. The main reasons are twofold. First, wave theory is traditionally presented in the context of particular physical objects, such as optical waves, radiowaves, plasma waves, waves in fluids, and acoustic waves. But mathematically, wave theory can be constructed and presented irrespective of the physical nature of the wave process. And second, the very important subjects of modern natural sciences are nonequilibrium physical systems, for which wave theory plays a secondary role and is merely part of such original branches of physics as physical kinetics, plasma physics, microwave electronics, to name but a few. Yet, there is clearly a need for an original theory of waves in nonequilibrium media. In our monograph, the general theory of linear waves is presented just as a branch of mathematical physics that describes the dynamics of linear waves in equilibrium and nonequilibrium dispersive media and systems, irrespective of their physical nature. The practical application of the general theory is illustrated by considering fairly simple but important physical systems that are traditionally studied in the mechanics of continuous media, electrodynamics, plasma physics, and electronics. The practical problems are solved by a unified approach presented in the mathematical part of the wave theory. Along with traditional information, the monograph contains a number of new original results that we have obtained in studying nonequilibrium and resonant phenomena in plasmalike media and that have found practical applications in electronics and radiophysics. In studying linear waves in dispersive media and systems, we proceed from the general to the special, and we hope that our theoretical study will be of interest to both beginners (students and postgraduates) and experts in the physics of wave processes. Note finally that the monograph is based on the course of lectures given by the authors to senior students at Moscow State University (Faculty of Physics, Division of Physical Electronics).

Contents

Chapter 1

Linear Harmonic Waves in Dispersive Systems. Initial-Value Problem and Problem with an External Source

1. Harmonic Waves in Dispersive Systems

Assume that small perturbations of the equilibrium state of a one-dimensional physical system satisfy the following set of first-order linear homogeneous partial differential equations:

$$\frac{\partial \psi_s}{\partial t} + \sum_{j=1}^{n} \left(A_{sj} \frac{\partial \psi_j}{\partial z} + B_{sj} \psi_j \right) = 0, \quad s = 1, 2, \ldots, n. \tag{1.1}$$

Here, $\{\psi_1(t,z), \psi_2(t,z), \ldots, \psi_n(t,z)\} \equiv \mathbf{\Psi}(t,z)$ is the vector of small perturbations of the equilibrium, called the state vector of the system. The number of components $\psi_s(t,z)$ of the state vector is equal to the number n of equations in set (1.1), and A_{sj} and B_{sj} are square $n \times n$ matrices with constant elements. Among the equations that are reduced to equations of the form (1.1), we can mention acoustic equations, hydrodynamic equations, equations for the electromagnetic field in various material media, linear plasma electrodynamic equations, linearized equations of theoretical microwave electronics, and many others.

We seek a solution to Eqs. (1.1) in the form

$$\mathbf{\Psi}(t,z) = \mathbf{\Phi}(\omega,k) \exp(-i\omega t + ikz)\,, \tag{1.2a}$$

$$\mathbf{\Phi}(\omega,k) = \{\phi_1(\omega,k), \phi_2(\omega,k), \ldots, \phi_n(\omega,k)\}\,. \tag{1.2b}$$

Here, $\mathbf{\Phi}(\omega,k)$ is the complex state vector; $\phi_s(\omega,k)$, with $s = 1, 2, \ldots, n$, are the state vector components; ω is the frequency; and k is the wavenumber. The state vector $\mathbf{\Psi}(t,z)$ is a physical quantity and as such is real. Consequently, only the real part of complex function (1.2a) has a physical meaning. It is convenient, however, to perform linear operations on the state vector in its complex form without restriction and to switch to its real part only in the final result.

Each component of the state vector (1.2a) is a plane harmonic wave, which is characterized by the time period

$$T = \frac{2\pi}{\omega} \tag{1.3}$$

and the spatial period, or wavelength,

$$\lambda = \frac{2\pi}{k} \,. \tag{1.4}$$

An important parameter of a plane harmonic wave is its phase velocity, i.e., the propagation velocity of the constant-phase points (planes) in space. This is the velocity at which an "observer" should move along the z axis in order for the state vector (1.2) to be constant. The phase velocity is obviously determined from the relationship

$$\omega t - kz = \text{const} \,, \tag{1.5}$$

which indicates that the phase of a plane wave is constant. Differentiating relationship (1.5) with respect to time and taking into account the fact that the observer's speed is dz/dt yields the definition of the phase velocity:

$$V_{ph} = \frac{\omega}{k} \,. \tag{1.6}$$

We substitute solution (1.2) into homogeneous equations (1.1) to arrive at the following set of linear homogeneous algebraic equations for the components $\phi_s(\omega, k)$ of the complex state vector:

$$-i\omega\phi_s(\omega, k) + \sum_{j=1}^{n}(ikA_{sj} + B_{sj})\phi_j(\omega, k) = 0, \quad s = 1, 2, \ldots, n \,. \tag{1.7}$$

The number of equations in set (1.7) and the number of unknowns $\phi_s(\omega, k)$ are both equal to n. Of course, we are interested only in nontrivial solutions to Eqs. (1.7), i.e., in such sets of state vector components $\phi_1(\omega, k)$, $\phi_2(\omega, k), \ldots, \phi_n(\omega, k)$ in which at least one is nonzero. Otherwise, the state vector (1.2) would be identically zero, a physically uninteresting case. From linear algebra, it is known that a set of linear homogeneous algebraic equations has a nontrivial solution if and only if its determinant is zero. For the set of Eqs. (1.7), this condition can be written as

$$D(\omega, k) \equiv \det(-i\omega\delta_{sj} + ikA_{sj} + B_{sj}) = 0, \quad s, j = 1, 2, \ldots, n \,, \tag{1.8}$$

where δ_{sj} is the Kronecker symbol. Relationship (1.8) is called the dispersion (characteristic) relation for determining the spectra of eigenmodes. The function of two variables $D(\omega, k)$ is called the dispersion function.

Dispersion relation (1.8) is a relationship between two independent quantities — frequency ω and wavenumber k. Consequently, this dispersion relation can be solved either with respect to frequency (in order to determine the dependence $\omega = \omega(k)$) or with respect to wavenumber (in order to find the function $k = k(\omega)$). The first approach yields a solution to the so-called initial-value problem. The second approach is used to solve the boundary-value problem. In the present monograph, we will only consider initial-value problems in which dispersion relation (1.8) is solved with respect to frequency and the frequency spectra of the eigenmodes, $\omega = \omega(k)$, are determined.

Dispersion relation (1.8) usually has more than one solution, i.e., $\omega = \omega_m(k)$, with $m = 1, 2, \ldots$. In this case, a physical system is said to have several branches of eigenmodes with eigenfrequencies $\omega_m(k)$. From Eqs. (1.1) and (1.7) we can see that dispersion relation (1.8) is an nth order algebraic equation for the frequency ω. In algebra courses, it is proved that such an equation has n roots, each corresponding to its own branch of eigenmodes. Hence, the number of different solutions to dispersion relation (1.8), or equivalently the number of different branches of eigenmodes, is equal to n. But it should be noted that some of the solutions to dispersion relation (1.8) can be trivial ($\omega_m = 0$) and therefore should be excluded from consideration,[1] in which case the number of eigenmode branches is in fact less than the number of equations in set (1.1). In addition, the dispersion relation can have coincident (multiple) roots — a so-called degenerate case that requires a separate analysis.

Corresponding to each eigenfrequency $\omega_m(k)$ there is a state eigenvector $\mathbf{\Psi}_m(t, z)$. The complex state eigenvector $\mathbf{\Phi}_m(\omega, k)$ that corresponds to the vector $\mathbf{\Psi}_m(t, z)$ is found from the set of algebraic equations (1.7) but with the eigenfrequency in place of an arbitrary frequency ω. In this case, the frequency ω is no longer an independent variable, so, in expression (1.2b), we can introduce the notation

$$\begin{aligned} \mathbf{\Phi}_m(\omega, k) &= \mathbf{\Phi}(\omega_m(k), k) \equiv \mathbf{\Phi}_m(k)\,, \\ \phi_s(\omega, k) &= \phi_s(\omega_m(k), k) \equiv \phi_s^{(m)}(k)\,, \\ \mathbf{\Phi}_m(k) &= \{\phi_1^{(m)}(k), \phi_2^{(m)}(k), \ldots, \phi_n^{(m)}(k)\} \equiv \{\phi_1(k), \phi_2(k), \ldots, \phi_n(k)\})(m)\,. \end{aligned} \tag{1.9}$$

Since the solution to a set of homogeneous algebraic equations is defined to within a constant factor, the vector $A_m \mathbf{\Phi}_m(k)$, with A_m being an arbitrary constant, also satisfies the set of Eqs. (1.7). Hence, in solving the initial-value problem, the state vector of a physical system that is described by linear differential equations (1.1) has the form

$$\mathbf{\Psi}_m(t, z) = A_m \mathbf{\Phi}_m(k) \exp[-i\omega_m(k)t + ikz]\,. \tag{1.10}$$

Moreover, there are as many such vectors as there are eigenmode branches, i.e., $m = 1, 2, \ldots, n$. And finally, keeping in mind the superposition principle, which implies in particular that a sum of solutions to a linear equation is also its solution, we write the solution to the initial-value problem for a set of linear homogeneous differential equations (1.1) as the sum over all eigenmode branches:

$$\mathbf{\Psi}(t, z) = \sum_{m=1}^{n} \mathbf{\Psi}_m(t, z) = \sum_{m=1}^{n} A_m \mathbf{\Phi}_m(k) \exp[-i\omega_m(k)t + ikz]\,. \tag{1.11}$$

Solution (1.11) contains the wavenumber k and constant factors A_m, which are called complex amplitudes. In order to determine the wavenumber and amplitudes,

[1] This concerns only harmonic waves; on the other hand, such solutions correspond to constant, but spatially nonuniform, fields.

additional conditions are required. How to formulate these additional conditions and how to use them will be described in Sec. 2.

Let us consider the phase velocity (1.6) for a particular harmonic eigenmode (1.10) of a certain physical system:

$$V_{ph}^{(m)} = \frac{\omega_m(k)}{k}. \tag{1.12}$$

If the phase velocity (1.12) is independent of the wavenumber k, then, according to the terminology adopted in the theory of linear waves, the eigenmode is said to have no dispersion. If the phase velocity $V_{ph}^{(m)}$ is a function of the wavenumber k, the eigenmode is called dispersive. Systems (media) in which there are dispersive eigenmodes are referred to as systems with dispersion or dispersive systems (media). For purely harmonic waves, the notion of dispersion is meaningless. But for more complicated, nonharmonic wave formations, the notion of wave dispersion plays an important role.

In accordance with what was said above, the eigenmode is nondispersive if its eigenfrequency is given by the formula

$$\omega_m(k) = \alpha k, \quad \alpha = \text{const} \tag{1.13}$$

Indeed, the phase velocity (1.12) in this case is independent of the wavenumber k. In wave theory, frequency spectra of the form (1.13) are called acoustic-like spectra.

Historically, the notion of dispersion has come to wave theory from optics. Since we are dealing with waves of quite a general nature, we extend the notion of dispersion as follows. A wave is considered to be nondispersive if its eigenfrequency has the form

$$\omega_m(k) = \alpha k + \beta, \quad \alpha = \text{const}, \quad \beta = \text{const} \tag{1.14}$$

For $\alpha = 0$, spectrum (1.14) is called optical. For $\beta \neq 0$, the phase velocity of a wave with the frequency (1.14) depends on the wavenumber k. But from the standpoint of the dynamics of nonharmonic wave formations, the frequency spectra (1.13) and (1.14) are equivalent, as will be shown later. We stress that, in spectra (1.13) and (1.14), the symbol "const" implies independence on the wavenumber k. For

$$\frac{d^2\omega_m(k)}{dk^2} \neq 0, \tag{1.15}$$

the eigenfrequency cannot be represented in the form (1.14) and the wave is dispersive. Inequality (1.15) is a mathematical criterion of whether the wave is dispersive or not.

Spatially harmonic solution (1.11) to the initial-value problem contains important information about the state of a physical system (medium). The solutions to dispersion relation (1.8) are generally complex,

$$\omega_m(k) = \omega_m'(k) + i\omega_m''(k), \tag{1.16}$$

so it is convenient to rewrite solution (1.11) as

$$\Psi(t,z) = \sum_{m=1}^{n} \Psi_m(t,z) = \sum_{m=1}^{n} A_m \Phi_m(k) \exp[\omega_m''(k)t] \exp[-i\omega_m'(k)t + ikz] \,. \quad (1.17)$$

If, for all m (i.e., for all the branches of eigenmodes), the imaginary parts are negative, $\omega_m''(k) < 0$, then all the terms in solution (1.17) decrease exponentially with time t. In this case, the negative imaginary part of the frequency is called the damping rate of the wave. On sufficiently long time scales, only the term with the minimum absolute value of the damping rate is important in solution (1.17). If one of the roots of the dispersion relation has a zero imaginary part, $\omega_m''(k) = 0$, then the corresponding term of the sum in solution (1.17) is not damped with time and describes an undamped eigenmode. And finally, if at least one of the roots has a positive imaginary part, $\omega_m''(k) > 0$, then the corresponding eigenmode grows with time. This is the case only when the system (medium) is in an unstable nonequilibrium state. The positive imaginary part of the frequency is called the growth rate of the wave or the instability growth rate.

In what follows, we will primarily focus on systems for which dispersion relations (1.8) are algebraic equations with real coefficients (except in Secs. 6–8, 10, 12, 17, 25). It is known that, if a certain complex number $\omega' + i\omega''$ is a root of such an equation, then its complex conjugate, $\omega' - i\omega''$, is a root too. It is also known that an algebraic equation with real coefficients can have no roots at all. That is, either we have $\omega_m''(k) = 0$ for all m, in which case the system is in a stable state, or there is an eigenmode branch such that $\omega_m''(k) > 0$, in which case the system is unstable. But it is somewhat incorrect to speak of wave damping in systems described by dispersion relations with real coefficients. Indeed, for any eigenmode branch with $\omega_{m_1}'' < 0$, there is a complex-conjugate branch with $\omega_{m_2}'' = -\omega_{m_1}'' > 0$ — a circumstance implying that the system is unstable. In actuality, wave damping always results from the dissipation of perturbation energy. A dispersion relation with real coefficients describes a nondissipative system.

Conceiving the wave phase velocity as the propagation velocity of constant-phase (but not constant-amplitude) points is also meaningful for complex frequencies. It is only necessary to rewrite formula (1.12) as

$$V_{ph}^{(m)} = \frac{\mathrm{Re}\,\omega_m(k)}{k} = \frac{\omega_m'(k)}{k} \,. \qquad (1.18)$$

It is also obvious that introducing the notion of the wave period (1.3) is meaningful only when the imaginary part of the frequency is much less than its real part.

2. Initial-Value Problem. Eigenmode Method

In order to complete an investigation of the problem of excitation of harmonic eigenmodes in a system described by differential equations (1.1), it is necessary to find the wavenumber k and constant complex amplitudes A_m in the general solution

(1.11). To do this in the most illustrative way, it is convenient to change the notation system, i.e., to pass over from row vectors (1.2) to column vectors. Thus, we write the harmonic solution (1.11) to Eqs. (1.1) as (see also (1.9))

$$\boldsymbol{\Psi}(t,z)=\sum_{m=1}^{n}\boldsymbol{\Psi}_m(t,z)=\sum_{m=1}^{n}A_m\begin{pmatrix}\phi_1^{(m)}(k)\\ \phi_2^{(m)}(k)\\ \vdots\\ \phi_n^{(m)}(k)\end{pmatrix}\exp[-i\omega_m(k)t+ikz]\,. \qquad (2.1)$$

Let us consider the structure of the column vector in solution (2.1) in more detail. The components $\phi_s^{(m)}(k)$ of the complex state vector satisfy the set of algebraic equations (1.7) with $\omega=\omega_m(k)$. Since Eqs. (1.7) are homogeneous, the components $\phi_s^{(m)}(k)$ can be found in the following way. The terms containing one of the components, say $\phi_1^{(m)}$ for definiteness, are moved to the right-hand side of Eqs. (1.7) and are treated as being known. The set of Eqs. (1.7) is then solved in a conventional manner (by linear algebra methods) for the remaining components $\phi_2^{(m)},\phi_3^{(m)},\ldots,\phi_n^{(m)}$. The resulting solutions are linear in $\phi_1^{(m)}$:

$$\phi_s^{(m)}(k)=L_s(\omega_m(k),k)\cdot\phi_1^{(m)}(k)\equiv L_{sm}(k)\phi_1^{(m)}(k),\quad s=2,3,\ldots,n\,, \qquad (2.2)$$

where L_s are functions of the coefficients of Eqs. (1.7). As for the components $\phi_1^{(m)}$, they are arbitrary and can be chosen to be, e.g., unity. In so doing, the dimension should be accounted for as follows. When the complex component $\phi_1^{(m)}$ of the state vector is dimensional, it is convenient to assign its dimension to the complex amplitudes A_m, i.e., in effect, to make the redefinition $A_m\phi_1^{(m)}(k)\equiv A_m(k)$.

The last point deserves some clarification. After the terms with $\phi_1^{(m)}$ have been moved to the right-hand side of Eqs. (1.7), the number of unknowns becomes $n-1$, while the number of equations remains equal to n — a situation that poses no mathematical difficulty, however. Since ω_m is a root of dispersion relation (1.8), the determinant of the set of Eqs. (1.7) is zero. Consequently, one (any one) of the equations is a consequence of the remaining equations and thus can be dropped from the set. Hence, the number of unknowns and the number of equations are in fact the same. An approach for finding the complex amplitudes $A_m(k)=A_m\phi_1^{(m)}(k)$ and the functions $\phi_s^{(m)}(k)$ (2.2) that is presented below is called the eigenmode method.

Assume that, at the initial time $t=0$, the spatially harmonic state vector of the system is given by the formula

$$\boldsymbol{\Psi}(0,Z)=\begin{pmatrix}b_1(\chi)\\ b_2(\chi)\\ \vdots\\ b_n(\chi)\end{pmatrix}\exp(i\chi z)\,, \qquad (2.3)$$

where χ and $b_s(\chi)(s=1,2,\ldots,n)$ are known (prescribed) constant quantities. Vector relationship (2.3) is an initial condition for differential equations (1.1). Specifically, Eqs. (1.1) supplemented with relationships (2.3) constitute a so-called initial-value problem or a problem with initial conditions. The problem at hand is an initial-value problem with harmonic initial conditions. Let us consider the main steps in finding its solution.

At subsequent times ($t > 0$), the state vector satisfies Eqs. (1.1) and is therefore described by formula (2.1) (or (1.11)). Substituting $t = 0$ into formula (2.1) and equating the result to the initial state vector (2.3) yields the relationship

$$\sum_{m=1}^{n} A_m \begin{pmatrix} \phi_1^{(m)}(k) \\ \phi_2^{(m)}(k) \\ \vdots \\ \phi_n^{(m)}(k) \end{pmatrix} \exp(ikz) = \begin{pmatrix} b_1(\chi) \\ b_2(\chi) \\ \vdots \\ b_n(\chi) \end{pmatrix} \exp(i\chi z) , \tag{2.4}$$

which should be satisfied identically for any $z \in (-\infty, +\infty)$. This is clearly the case only when $k = \chi$. Hence, the wavenumber k in solution (2.1) (and in (1.11)) is determined by the structure of the initial perturbation of the state vector that is harmonic in the spatial variable z. The case of a nonharmonic perturbation will be considered below.

Taking into account the equality $k = \chi$ and cancelling the common exponential factor in relationship (2.4), we obtain the set of linear algebraic equations

$$\sum_{m=1}^{n} A_m \begin{pmatrix} \phi_1^{(m)}(k) \\ \phi_2^{(m)}(k) \\ \vdots \\ \phi_n^{(m)}(k) \end{pmatrix} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix} , \tag{2.5a}$$

in which, by virtue of relationships (2.2), the complex state vector components $\phi_s^{(m)}$ are knowns. From the set of Eqs. (2.5a) we can determine the unknown complex amplitudes $A_m = A_m(k)$.

With relationships (2.2), we introduce the new notation $A_m \phi_1^{(m)}(k) \equiv A_m(k)$ to rewrite Eqs. (2.5a) as

$$\sum_{m=1}^{n} A_m(k) \begin{pmatrix} 1 \\ L_2(\omega_m(k), k) \\ \vdots \\ L_n(\omega_m(k), k) \end{pmatrix} \equiv \sum_{m=1}^{n} A_m(k) \begin{pmatrix} L_{1m} \\ L_{2m} \\ \vdots \\ L_{nm} \end{pmatrix} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix} , \quad L_{1m} \equiv 1 . \tag{2.5b}$$

It is in this form that the equations are usually used to solve particular initial-value problems.

Concerning the set of Eqs. (2.5), some points need to be clarified. If the number of equations in set (2.5) is equal to the number of unknowns, then the equations can be solved unambiguously by linear algebra methods. The number of equations is equal to the number n of state vector components, and the number of unknowns is equal to the number of wave branches, i.e., to the number of solutions to dispersion