Construction and Applications of Conformal Maps

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Preface

The development of calculating machines, which showed sturdy growth during the first part of the present century, has recently received a startling stimulus with the completion and successful operation of several fully automatic, electronically sequenced, general purpose digital machines capable of performing at fantastically high speeds.

In recognition of the need for basic research and training in the types of mathematics which are pertinent to the efficient exploitation and further development of such machines, the National Bureau of Standards established the Institute for Numerical Analysis, as a section of the National Applied Mathematics Laboratories, in 1948 on the campus of the University of California, Los Angeles. In this endeavor, the Bureau operates with the cooperation of the University, and with the support of the Office of Naval Research of the Navy Department.

At the end of its first year of existence, the Institute was host to two symposia, one on conformal mapping and the other on the Monte Carlo method. The papers in the present volume were prepared for presentation at the conformal mapping symposium.

Conformal mapping was chosen as a symposium topic for several reasons. This is a very old and classic branch of mathematics, going back even to Ptolemy. There are many existence theorems, starting with Riemann's, yet little attention has been given to constructive details. The differential equations involved are far from trivial, but they are not so complicated as to preclude the possibility of their being handled by machines now coming into use. Many excellent mathematicians are interested in the field, and it was felt that the symposium would serve well to introduce them to, and interest them in, the use of high-speed machines for the approximate solutions of mathematical problems. Finally, there are many important physical applications of conformal mapping.

I deeply appreciate the encouragement given by John H. Curtiss, chief of the National Applied Mathematics Laboratories, and by Hugh Odishaw, assistant to the Director of the National Bureau of Standards, during the organization of the symposium and the preparation of this volume; and the help given by Cornelius Lanczos, Alexander Ostrowski, and Wladimir Seidel, who served with me as a committee of the Institute in arranging the symposium. The contributors to this volume also have been most helpful in promptly submitting manuscripts.

E. F. BECKENBACH, *Editor*.

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Contents

1.   On Network Methods in Conformal Mapping and in Related Problems [1]

R. v. Mises [2]

A fairly general type of problem of mathematical physics can be written in the form

$$L(u)=0, \quad B(u)=0,$$

where L is some differential operator in n dimensions, and $B(u)=0$ stands for a set of so-called boundary conditions, that is, of equations for u and its derivatives valid in subspaces of less than n dimensions. As the simplest case, one may consider the ordinary problem of conformal mapping: $L(u)=\Delta(u)$, $n=2$ [$B(u)=0$ denoting given u-values on a given contour]. There are, however, much more complicated questions of interest today to physicists and engineers. To characterize the subject matter envisaged in the present address the following four examples may be quoted, two of which belong to the theory of conformal mapping in a wider sense.

(a) The steady two-dimensional potential flow of an incompressible fluid past a weir requires the solution of $\Delta u=0$ with a free boundary of unknown shape, along which the velocity is proportional to the square root of the variable depth. A first approach to an adequate solution was given in the Berlin thesis of Lauck [1],[3] but no further investigation followed this preliminary attempt.

(b) The Helmholtz flow with separation past a closed (for example, circular) obstacle has been studied by many authors. It is, however, not yet known for sure whether or not the conjecture advanced by C. Schmieden [2] that a definite continuous range of separation points exists for certain kinds of obstacles is correct. The answer to this question would be most important for our general understanding of fluid motion (boundary layer theory, and so on).

(c) The efflux of a perfect fluid through an arbitrarily shaped (for example, square) orifice in the bottom of a tank presents a problem of three-dimensional potential theory. Nothing has been done so far about the solution. It is not even known whether the conspicuous changes in the cross section of the jet [3] can be explained on the assumption of a constant surface velocity or whether they are due to the influence of gravity or, possibly, of capillarity.

(d) A great deal of mathematical work is being devoted at present to the study of the two-dimensional steady flow of a compressible fluid. However, not a single case of flow past a body of given shape has been solved in an acceptable way, not even for subsonic or shockfree supersonic motion. The situation is still more involved in the case of a flow with a curved shockfront, behind which a modified differential equation has to be considered.

These examples may suffice to show what the state of affairs actually is and to what extent the help of mathematicians is needed for the progress of physics and engineering. If one looks up the published literature in related mathematical fields as it has developed in the last two centuries, he finds that the overwhelming majority of studies does not deal with any numerically specified boundary value problems at all. What the mathematicians did—and nobody will deny that they were right to do so—was to explore general properties of solutions and to determine particular integrals of the differential equations without direct reference to given boundary conditions. A quite considerable and still growing body of knowledge has been accumulated in this way, which is most helpful in various respects to the physicist and to the engineer. This is particularly true in the realm of potential theory and of conformal mapping which gave rise to entire branches of mathematical analysis. But there are also problems in which nothing short of an explicit numerical solution can satisfy the actual need. Moreover, as mentioned above, in many instances highly important principal questions cannot be answered on the basis of existing theories, and the only way to obtain some insight into the complicated relationships seems to be the search for numerical results. Up to date, mathematicians are still busy remodeling and refining the methods of computation in the simplest problems of conformal mapping. Should we wait with all other questions until an analogous state of perfection is reached in all pertinent fields?

The usual mathematician's approach to a specified boundary value problem runs as follows: One starts from an existence and unicity theorem that is either mathematically proved or assumed on the basis of experimental evidence. It is stated that one and only one point function $u(P)$ exists for which $L(u)=0$ and $B(u)=0$. Then, some computational process, depending on an integer n, is devised that leads to certain values $u^{(n)}(P)$ for all points P of the domain under consideration. Finally, if it can be proved that for all these points uniformly

$$\lim_{n=\infty} u^{(n)}(P)=u(P),$$

then it is said that a solution has been found. The idea, obviously, is that if n is sufficiently large, the difference $|u^{(n)}(P)-u(P)|$ will be so small as to have no significance in consideration of the unavoidable inaccuracies of the theory implied in the functions L and B. It is well known that in many cases the procedure leading to $u^{(n)}$ is so complicated that very low numbers n only can be used. There is a widely spread feeling among mathematicians—sometimes justified, sometimes not—that once the convergence of a sequence of operations has been proved, the first or the second member of the sequence must be somehow "good."

It is obvious that this shortcoming could be overcome only if a reasonable upper bound of the error $|u^{(n)}(P)-u(P)|$ were known for each n. In very rare cases such an estimate of the error magnitude is available. Often the estimation, if it is possible at all, is based on more or less vague probability assumptions, and almost always its application requires extended computational work. It must also be noted that the relation between the inaccuracy of L and B and the admissible error margin $|u^{(n)}-u|$ cannot be determined, in general, in a rational way.

As to the types of convergent processes to be used in solving a boundary-value problem, there are two schools of thought that diametrically oppose each other. The more traditional methods consist of series developments and of so-called successive approximations. It is generally assumed that the set of formulas describing the computational process in these cases offers some insight into the analytic properties of the solution. In strict opposition to these methods is the procedure that starts by assuming a network of points P_i and replaces the differential quotients in L (and, as the case may be, in B) by difference quotients, thus reducing the original analytic problem to the algebraic problem of solving a set of simultaneous algebraic equations. The underlying idea is that the values u_i, thus computed, will converge towards the values of $u(P_i)$ as the mesh size of the network is reduced by successive steps to zero. This "network method" is usually viewed askance by most mathematicians.

There is no logical principle that would allow us to distinguish between the two types of procedures. If we accept the point of view that convergence toward the exact solution is the only thing that matters, then any convergent computational process, unambiguously described by means of formulas or of colloquial language, is logically equivalent to any other. The convergence proof will, in general, be easier in the case of the network methods. A point for the analytic method can sometimes be made with respect to the dependency of the solution on variable parameters. However, if this dependency is of simple form (as, for example, similitude) it can also be incorporated in the network procedure. If it is very complicated, the parameters enter into the computation of the functions used in the development or in the successive approximations, and thus no practical advantage results. One should also not forget that all formulas except the simplest ones include some integrations or quadratures that can be performed only by means of interpolation formulas; this means, in fact, that the network point of view enters through the back door. One further point should be noted. It is often said that in applying the network method the original choice of the mesh size definitively determines the degree of approximation, and when a higher degree is to be attained the whole procedure must be repeated without any benefit drawn from the preceding work. This is not quite correct. In all problems of advanced character the main difficulty lies in the solution of the numerous simultaneous algebraic equations that can be performed by iterations only. Then the results obtained with a wider network supply the adequate point of departure for a second step of computation with finer meshes.

In summarizing this evaluation of the two ways of approach, we must say that the case for neither one can be made on general terms. The merits and the demerits must be weighed in the light of the particular circumstances of each single problem. One might take for granted that in the simplest cases like that of conformal mapping with fixed, not too complicated boundaries, the analytic way will be preferable or, at least, equally accessible. In the theory of gas flow where the differential equation is much more involved and the boundaries are given in the physical plane while the integration is carried out in the hodograph plane, so far not a single problem could be solved satisfactorily by analytic methods, in spite of enormous and, certainly, very meritorious efforts. Perhaps here a combined method could be applied. The point to be stressed is that both ways of approach, the analytical and the network, are from a logical standpoint of equal rights. They both consist of a truncated convergent process without any practical possibility for estimating the degree of approximation reached after a finite number of steps.

The question arises as to whether this latter shortcoming cannot be avoided or at least lessened by a more realistic concept of the situation that presents itself when we are confronted with a numerically

specified boundary value problem in any field of mathematical physics. Everybody agrees that it makes little sense to ask for an exact, that is, precise solution in view of all the simplifying assumptions used in the mathematical formulation of the problem—for instance, neglecting the influence of friction, gravity, heat transfer, and so on. Why then start from the proved or presumed fact that an "exact" solution exists and look for a sequence of functions that converges towards this solution? Is it not more natural to admit that both relations $L(u)=0$ and $B(u)=0$ are only approximately valid, and that what we really have to look for is a function u for which

$$|L(u)| \leqq \epsilon \quad \text{and} \quad |B(u)| \leqq \epsilon?$$

This ϵ, at any rate, has a definite physical meaning. In $L(u)$ it is the allowance made for the influence of disregarded physical factors, such as friction; in $B(u)$ it is a measure of the accuracy of the given conditions. It makes sense to assume that the data of a problem in physics or engineering include a pertinent indication concerning the order of magnitude of ϵ.

In the course of a numerical computation a point function $u(P)$ can be determined only by a finite number of values u_i associated with certain points P_i $(i=1,2, \ldots , n)$ of the domain under consideration. Thus, if the above suggestion is accepted, the complete problem has to be formulated as follows. Given the expressions $L(u)$, $B(u)$, and a positive quantity ϵ, find n values $u_1, u_2, \ldots , u_n$ such that a function $u(P)$ exists that takes on these values at points $P_1, P_2, \ldots , P_n$, and satisfies the inequalities $|L(u)| \leqq \epsilon$, $|B(u)| \leqq \epsilon$. This formulation, which eliminates entirely the question of convergence, still leaves the question open as to whether an analytical development or the network point of view should be used. It is certainly not an easier problem that is laid before us in this way, but it seems to be the very problem to which the actual situation, as it presents itself to the physicist or engineer, leads. A few indications may be given to show how the modified question can be approached in case one decides in favor of the network point of view.

Let us assume that a Cartesian network with equal quadratic meshes is used and that $L(u)$ is a differential operator of second order,

$$L(u) = A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y},$$

where $A, B, \ldots, E$ depend on x, y, u. The substitutions to be made for the five derivatives are given by the familiar expressions ($h=$ mesh size):

$$\frac{\partial u}{\partial x} \sim \frac{u_{i+1,\,k} - u_{i-1,\,k}}{2h}, \ldots, \quad \frac{\partial^2 u}{\partial x^2} \sim \frac{u_{i+1,\,k} - 2u_{i,\,k} + u_{i-1,\,k}}{h^2}, \ldots, \quad \frac{\partial^2 u}{\partial x\,\partial y} \sim \frac{u_{i+1,\,k+1} - u_{i-1,\,k+1} - u_{i+1,\,k-1} + u_{i-1,\,k-1}}{4h^2}.$$

If no complications are caused by the boundary conditions, these expressions introduced in $L(u)=0$ lead to a system of n algebraic equations for the u_{ik}. Suppose these equations are solved. How can the deviation ϵ be determined?

One has, first, to construct a function $u(x,y)$ that assumes the computed values u_{ik} at the nodal points P_{ik} and that admits continuous first- and second-order derivatives for all x,y. It turns out that the function can be subject to the additional conditions that its derivatives up to the second order at all nodal points equal the values of the respective difference quotients at these points. This determines uniquely a function composed of eighth-order polynomials with coefficients varying from one mesh area to the next. If we consider the square formed by the points $P_{11}, P_{12}, P_{21}, P_{22}$, with the coordinates (x_1,y_1), (x_1,y_2), (x_2,y_1), (x_2,y_2), and write

$$\xi = \frac{x - x_1}{h} - \frac{1}{2}, \quad \eta = \frac{y - y_1}{h} - \frac{1}{2},$$

the function $u(x, y)$ within the square is given by

$$u(x, y) = \sum_{i,\,k}^{0 \ldots 3} \alpha_i(\xi)\,\alpha_k(\eta)\,u_i u_k; \qquad |\xi| \leqq \frac{1}{2}, \quad |\eta| \leqq \frac{1}{2},$$

where

$$\alpha_0(\xi) = \alpha_3(-\xi) = \left(\xi - \frac{1}{2}\right)^3 \left(\xi + \frac{1}{2}\right)(\xi + 1), \quad \alpha_1(\xi) = \alpha_2(-\xi) = \left(\frac{1}{2} - \xi\right)\left[1 + \left(\frac{1}{2} + \xi\right)\left(\frac{1}{4} + \frac{9}{4}\xi - 3\xi^3\right)\right].$$

In the same way $u(x,y)$ is obtained for each other mesh. It can easily be shown that this $u(x,y)$ has continuous derivatives up to the second order along the lines that separate the meshes. The value of u inside a mesh depends on the 16 u_{ik}-values at the surrounding 16 nodal points. Thus, the deriva-

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tives and, finally, the value of $L(u)$ for any point can be computed as a linear expression in the 16 u_{ik} with coefficients depending on the coordinates ξ,η and the five factors A, B, C, D, E. By maximizing with respect to ξ,η, we find ϵ as a linear function of the 16 u_{ik}, which can be rearranged as a linear form in the third-order differences determined by the u_{ik}-values. At all nodal points the value of $L(u)$ is precisely zero. If, in some quadrangles, ϵ turns out to be too large, this is an indication that a smaller mesh size has to be chosen, either for the whole domain or for a part of it.

Difficulties of many kinds can arise in connection with the boundary conditions. In particular, any discontinuity or other singularity at the borderline may interrupt the smooth working of the method. This is the point where analytic considerations have to be made, and sometimes elaborate formulas, developments, and so on, will have to be used. One should not think of the network method as a whole as something quite elementary, not requiring any mathematical skill. Worthwhile results will be achieved only through an intensive cooperation of both points of view. It is possible that in the future most or all of the results attained in analytic studies of differential equations will have to be utilized as auxiliary in supplementing the computational procedure based on the network idea. Mathematicians should not entirely neglect this line of thought.

Even if no trouble comes from the boundary conditions, there are numerous questions that have to be studied when one wishes to apply the network procedure in a rational way. In the first place, it is by no means obvious that a Cartesian net is the right answer in all cases. The differential equation, the shape of the domain, and so on, may suggest a different choice, for example, a polar net consisting of radii and concentric circles. Very little is known about the possibility of replacing derivatives by difference quotients in a general network. Here are some simple results. If a second-order operator in two variables,

$$L(u) = A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2},$$

has to be approximated at a point P_0, then, in general, five (not four) neighboring points $P_0, P_1, \ldots, P_4$ must be used. The number reduces to four in special cases only. The condition is that the five points $P_0, P_1, \ldots, P_4$ lie on a conical curve whose equation with respect to P_0 as origin is

$$\alpha x^2 + 2\beta xy + \gamma y^2 = 0,$$

whereby

$$A\alpha + 2B\beta + C\gamma = 0;$$

that is, the scalar product of the two symmetrical tensors A, B, C and α, β, γ must vanish. In the case of the Laplace equation, $A = C$, $B = 0$, the associated conics are all equilateral hyperbolas, including the pairs of orthogonal straight lines. It follows that sets of parallels to the x- and y-axes are always admissible even if boundary points do not coincide with the crossing points. Likewise, a polar system with equidistant radii is admissible. The coefficients in the linear equations can easily be determined in each case. In the hyperbolic problem, $A = C = 0$, the associated conics are those with principal axes in the x- and y-directions, including pairs of symmetric straight lines. Here, the choice is very much restricted and, for example, a polar system is not applicable.

Another question that occasionally has been discussed in the literature is that of so-called higher-order approximations. Instead of substituting for Δu the difference between u_0 and the arithmetical mean of the four values u_1, u_2, u_3, u_4, at the neighboring cross points, one may introduce more nodal points in a Cartesian net and thus arrive at an error proportional to a power higher than the third of the mesh width h. A detailed study of such formulas has been given by L. Collatz in his Berlin thesis [4]. This approach is adequate in the case of a linear, homogeneous differential equation. If, however, the factors A, B, C depend on u, the advantage of a higher-order approximation in the derivatives becomes illusory. This can be seen if one considers the ordinary differential equation $y' = f(x,y)$. Here, a better approximation cannot be achieved by substituting for y' not $(y_{i+1} - y_i)/h$, but an expression involving more y-values. It is well known that the right thing to do is to apply the principles of interpolation to the right-hand member $f(x,y)$. The same idea must be followed in the case of partial differential equations. Unfortunately the interpolation theory for more than one independent variable is not very much developed. An alternative way to higher approximation formulas is indicated by the application of the calculus of variations as has been shown in several papers by R. Courant, K. Friedrichs, and H. Lewy [5]. These papers are almost the only ones in existence offering a more profound study of the network method.

The most extensive rôle in all computational projects dealing with integration problems is played by the necessity of solving a large number of simultaneous algebraic equations, linear or nonlinear. Except for some rare cases of pure initial value (Cauchy) problems of hyperbolic equations, this task presents itself not only if the network method is used but also in various analytical procedures, in development in series of orthogonal functions, in the Rayleigh-Ritz method, and so on. Sometimes,

in elliptic problems, analytic methods lead to a reduction of the number of equations from the order of magnitude n^2 to the order of magnitude n. This advantage, however, is in general outweighed by the complications involved in setting up the individual equations. In the hyperbolic case, with combined initial and boundary values, the number of simultaneous equations is already of the order n (linear subdivision of the integration domain). It seems that here lies the most promising field for the application of the network procedure, since the complications connected with an adequate arrangement of the equations do not occur.

Once the set of numerical equations has been established, the remaining task must be considered as a matter of mechanical computation. There is no doubt that the only method of solution for large sets is the method of iterations (successive approximations). The question of the convergence of iterations in the linear case has been clarified to a high degree and, if so desired, one can always use the Gauss transformation, which carries any set of linear equations into one with a positive definite matrix. The actual carrying out of the iterations, in the case of 500 to 1,000 and more simultaneous equations, can be done by big automatic computing machinery only. From all that is known about existing machines and those under construction, none of them is so far up to this task. One always learns how many thousands of multiplications per second one machine or the other can perform. But the time must come when the efficiency of a computing device is not measured in these terms, but by stating how many simultaneous equations it can solve in 1 day or in 100 or 200 hours. To the layman it is hard to understand why at least a semiautomatic device for this purpose could not be built, that would, for example, feed continuously prearranged sets of coefficients into a mechanism able to store the variables and to do the multiplications and the summing up. In most cases the number of terms in each single equation is very narrowly restricted. If competent people feel that such a goal cannot be attained in the near future they should say so instead of stimulating unreasonable expectations to the effect that any numerical problem, however complicated, will soon be accessible to solution by present-day computing machines.

The deficiency of the existing computing machines is brought into the open by the undeniable success of the so-called relaxation technique which, by its very nature, is an expedient tool for the human and, one might say, intelligent computer. The fact is that the succession of equations and of variables successfully used in a convergent iteration process is to a large extent arbitrary, and that a judicious choice can materially shorten the whole process, as long as the number of variables is not too large. The guidance given to the computer by R. V. Southwell's idea of "relaxation" [6] has brought forth a number of interesting results and is likely to produce many more. However, a procedure that makes essential use of the computer's intuition cannot serve as the basis for large-scale computational work that we expect to have done one day by means of automatic or semiautomatic machines.

A few points emphasized in this paper may be summarized as follows:

1. Network methods must be considered as a fully legitimate and often preferable way for solving numerical boundary value problems. They do not differ in principle from the more traditional methods of series development, and so on, as long as the convergence point of view is accepted.

2. One may, in certain cases, abandon the convergence concept and adopt the idea of approximating both the differential equation and the boundary conditions. The network methods lend themselves particularly well to this point of view.

3. Various problems arising in connection with the application of network methods are worthy of the attention of mathematicians—for example, the treatment of singularities, the choice of the network, and higher order approximations.

4. Practical progress in the numerical solution of boundary value problems can be expected only if new computing devices are built that are suitable for solving large sets of simultaneous equations in an automatic or semiautomatic way.

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2. Flow Patterns and Conformal Mapping of Domains of Higher Topological Structure

R. Courant [1]

Much of Riemann's geometrical function theory was motivated by phenomena of flow; these phenomena have also been used experimentally to obtain approximate solutions for problems of conformal mapping. A two-dimensional flow pattern is generated in a suitable medium, for example in an electrolytic tank, by impressing electromotive forces on the medium. Such flow patterns, obtained under appropriate boundary conditions, can be measured; they furnish the immediate solutions for fundamental problems of conformal mapping.

The present paper is concerned with extensions of Riemann's mapping theorem to domains of arbitrary connectivity and moreover to domains not of genus zero, such as a torus, a Moebius strip, or the projective plane.

From a theoretical point of view, the problem is to find "classes $\mathfrak{N}$ of normal domains" in a u,v-plane depending on a finite number of parameters and representing by conformal equivalence arbitrary domains G of the same topological structure. For simply connected domains Riemann's mapping theorem states that such a set $\mathfrak{N}$ is the exterior of a straight slit in the plane, a slit that is permitted to shrink to a point in a limiting case. For higher connectivity k the simplest and most easily obtainable classes of normal domains are "parallel-slit domains" or, more briefly, "slit domains." They consist of the whole plane of the complex variable $w=u+iv$ except for straight segments $v=$ constant, the "boundary slits." (In the case of infinite connectivity, there are infinitely many boundary slits.) Slits that degenerate into points also are admitted.

Likewise, one may consider "half-plane slit domains," consisting of the half-plane $v>0$ (instead of the full plane) except for a number of parallel slits. The line $v=0$ is then one of the boundaries.

By the following physical argument, one is led to the conjecture that such k-fold connected slit domains constitute classes of normal domains. Suppose that G is a k-fold connected domain in the plane of the complex variable $z=x+iy$. Consider a potential flow in G coming from a dipole O at the interior point $z=0$. This flow may be characterized by an analytic function $w=u+iv=f(z)$ with the singularity $1/z$ at $z=0$. In this representation, the curves $u=$ constant are equipotential lines, and the curves $v=$ constant are streamlines of the flow. The boundary curves of G, which we may visualize as smooth, are parts of streamlines $v=$ constant. It is plausible that the streamlines are analytic closed curves through O, along which u varies monotonically from $-\infty$ to $+\infty$, with the exception of the k streamlines $v=c_1$, $v=c_2$, . . . , $v=c_k$, which reach the boundaries, splitting there into two branches, and passing around the boundary in different directions until they meet again to lead back to the source at 0, as in figure 2.1. Each streamline $v=c$, except for $c=c_1$, c_2, . . . , c_k, is then mapped biuniquely onto a full line $v=c$ in the w-plane. Of the streamlines $v=c_1$, $v=c_2$, . . . , $v=c_k$, the parts coinciding with the boundary are mapped onto slits, that is onto straight segments $v=c_1$, $v=c_2$, . . . , $v=c_k$ in the w-plane, in such a way that the two edges of the slit correspond to the two branches of the streamline along the corresponding boundary curve of G. If the dipole O is placed on a boundary curve of G with its axis in the direction of the curve, we obtain a half-plane slit domain. This domain may be chosen as the upper half-plane by mapping the boundary curve on the entire line $v=0$.

In the preceding consideration the choice of six real parameters for a given domain G is left open, since we may place the source at an arbitrary point z_0 of G, choose direction and intensity of the dipole arbitrarily, and modify the mapping function by addition of an arbitrary constant. In other words, we may specify a mapping function in the form

$$u+iv=w=f(z)=\frac{a}{z-z_0}+b+(z-z_0)R(z),$$

where $R(z)$ is regular and analytic in G, a and b are arbitrary complex parameters, and z_0 is an arbitrary point in G. Since a k-fold connected slit domain depends on $3k$ parameters, namely, the initial points and lengths of the k slits, we are led to the conjecture that for $k>2$ a reduction of the number

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of parameters to $3k-6$ is possible. By considering half-plane slit domains this number of "moduli" is more readily motivated. Since $v=0$ is a boundary line, only a real additive constant b is admissible, and z_0 and a are similarly restricted; thus only three parameters remain at our disposal for reducing
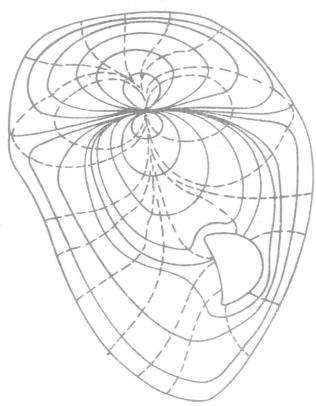

FIGURE 2.1.

to $3k-6$ the $3(k-1)=3k-3$ parameters in a k-fold half-plane slit domain. We may fix these parameters, for example, by making the point $w=\infty$ correspond to a given boundary point of G and by fixing the initial point of another slit as the point $w=i$.

Such mapping functions can be realized by actual experiments. It would seem that the experimental techniques developed in connection with electrolytic tanks can produce good results if it is assumed that the domain G contains the point at infinity, and if the dipole is placed at this point.

The mathematical construction of the mapping functions is given by Dirichlet's principle.

It may be added that instead of a dipole as the electromotive force, one may consider a number of simple poles, terminals of electric batteries. Somewhat different, although still not complicated, flow patterns and corresponding mapping theorems result.

Furthermore it is significant that the mapping theorems and their interpretation by physical experiment can be extended to domains G not in a plane but on polyhedra or curved surfaces in space. The same is true for any type of a Riemann domain, whether it is a plane or not.

These facts are well known if G is of genus zero. In the present paper we emphasize domains G of higher topological structure.
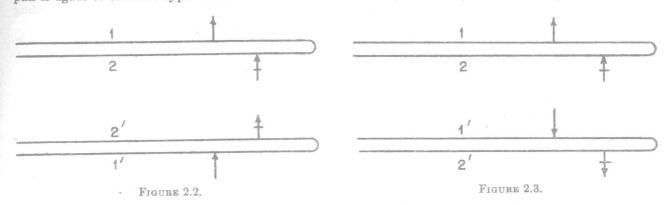
In this case, obviously a topological mapping—and therefore a conformal mapping—of G simply onto a plane domain is not possible. However, we again obtain slit domains as a class $\Re$ of normal domains. These slit domains consist of the whole w-plane with the exception of finite boundary slits parallel to the u-axis. In addition, the plane is cut by pairs of parallel slits $v=$constant which extend from $u=-\infty$ to a finite value of u and whose edges are coordinated in such a manner that the image slit domain B in the w-plane has the prescribed topological structure.

Description of slit domains not of genus zero. To define such plane slit domains, we first consider domains without boundary slits. There are two types of "interior" slit pairs: (1) a pair of slits $v=$ constant in the (u, v)-plane may extend from $u=-\infty$ to a finite value of u; we coordinate or identify the four edges as in figure 2.2, so that corresponding points have the same u-coordinate, and corresponding edges are marked with the same figures (*pair of the first type*); (2) a pair of slits which is similar except that the coordination is in figure 2.3 (*a pair of the second type*). These boundary coordinations imply

that paths meeting an edge of a slit have to be continued from the corresponding slit, as indicated in figures 2.2 and 2.3.

In order to have an admissible domain, at least one more pair of slits is necessary. If the second pair is again of the first type we obtain an orientable surface, the torus; if the second pair of slits is of

FIGURE 2.2.   FIGURE 2.3.

the second type, we obtain a nonorientable surface, the nonorientable torus or "Klein bottle." It should be remarked that in the second example the orientation of the surface is preserved along the closed path consisting of the slits of the second type and reversed along the slits of the first type.

A plane with just one pair of slits of the second type represents an admissible surface, the projective plane or cross cap, which may also be visualized as a disk where diametrically opposite points are identified.

A surface is characterized topologically by its orientability, its genus, and the number of its boundaries. Orientability depends on the absence of pairs of slits of the second type. Genus and number of boundaries are related to the total number of slits: Any surface can be considered topologically as a polyhedron having a certain number F of faces, E of edges, and V of vertices. The characteristic number L is defined as stipulated coordination.

The *condition that the point at infinity in the slit domain is an interior point* can be simply expressed by the following equivalent statement: A vertical line $u=$constant$=c$ that intersects all the slits is fully described if, on meeting an edge of a slit, one continues at the corresponding edge in the same or the opposite direction according as the pair is of the first or the second type, and proceeds in this manner until the path is closed. This condition implies a certain interlocking of the slits. The condition is not satisfied, for example, by a plane with just one pair of slits of the first type.

A domain with a pair of slits of the second type is nonorientable. This is immediately seen from figure 2.3, which shows that the boundary coordination between the slits reverses the orientation of an adjacent angle.

For a pair of slits of the first type, however, the orientation along such a closed path, as in figure 2.2, is that given by the slit coordination. Plane regions that contain only slit pairs of the first type are orientable.

We may cut the plane by k pairs of slits, with the edges coordinated as in the diagrams. Such a plane with coordinated slits need not have the topological structure of a closed surface. Although, by the identification of edges, any point on one of the infinite slits is obviously an interior point of our slit domain B, this will be true of the point at infinity only if the slits are properly arranged. If the slit domain B is to correspond topologically to a Riemann domain G, with the point at infinity corresponding to an interior point O, then a closed curve on G enclosing O will have to correspond to a single closed line in the slit domain which separates this domain into two parts. Such a line in the slit domain is made up of segments separated visually in the w-plane but connected by

$$L=2-F+E-V.$$

If G is orientable, of finite genus p, and is bounded by a finite number r of closed Jordan curves, then
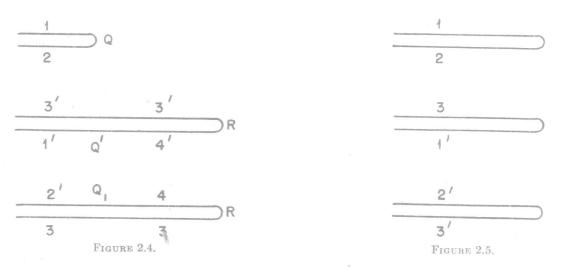
$$L=2p+r.$$

If G is nonorientable and is visually described by p "handles" and q "cross caps," then

$$L=2p+q+r.$$

For a closed surface that is represented by a plane slit domain B with s pairs of slits, we have $F=1$, $E=2s$ (corresponding edges being identified), and $V=s+1$, the point at infinity counting as a single

vertex.   Hence $s=L$.   Thus, *if the characteristic number of our slit domain is to be L, the domain must have exactly L pairs of infinite slits*, for example, $2p$ pairs of the first type slits if it is orientable and of genus p.
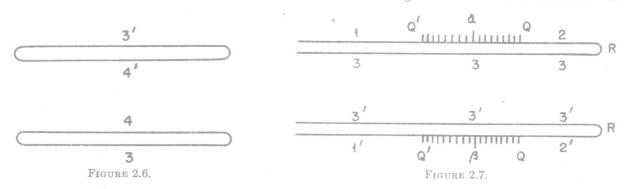
The class of slit domains B is now extended by including the limiting cases obtained if different pairs coincide entirely or in part.   As illustrations examine figures 2.4, 2.5, 2.6, where corresponding edges are indicated by the same number.

FIGURE 2.4.

FIGURE 2.5.

Similar situations arise when pairs of slits of the second type are present.

So far our slit domains B have been visualized as being closed.   We now consider r boundary slits. A boundary slit may be either a finite slit $v=$ constant—not coinciding with one of the infinite slits—or

FIGURE 2.6.

FIGURE 2.7.

it may coincide with a part of an infinite slit not containing the end-point, as illustrated by figure 2.7. In the latter case the coordination of the two edges of an infinite slit is interrupted along a finite segment (α and β in the diagram).   Otherwise, a boundary slit may at the same time form the end part of a pair of corresponding slits; in this case the end-points of the coordinated pair need not have the same u coordinate, and the coordination of the ends is interrupted, as in figure 2.8.   Here the segments α, β, γ, δ form the boundary slit under consideration.   In a similar way this may happen if the pair of infinite slits is of the second type or if, as in figure 2.9, the coordination embraces more than two slits.

As in the case for genus zero, we shall find it convenient to consider *half-plane slit domains* lying in the half-plane $v>0$, with $v=0$ corresponding to a boundary line; otherwise, nothing need be changed in our preceding description.   For example, a Moebius strip is represented by figure 2.10, and the limiting case where one of the infinite slits falls into $v=0$ is represented in figure 2.11, with the boundary indicated by shading.
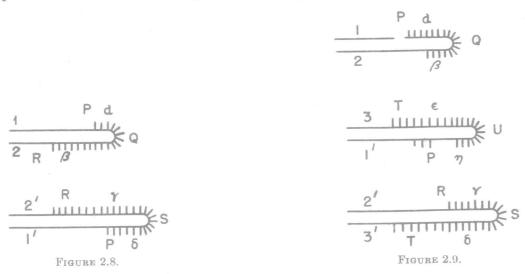
For such half-plane domains we may again stipulate a normalization similar to that used for genus zero: We may place the map of a fixed boundary point of G at the point at infinity on $v=0$ and fix the left end of one interior slit at $u=0$, $v=1$.

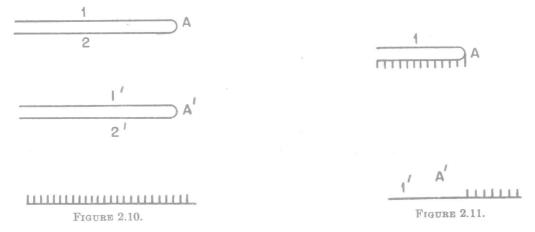*The mapping theorem.*   We now consider the class of all slit domains B with r such boundary slits and with characteristic number L, and state the following result.

Theorem: Every Riemann domain G, orientable or not, with characteristic number L and with r boundaries, can be mapped conformally on a slit domain B.   The mapping on a normalized half-plane slit domain is uniquely determined if the point at infinity is to correspond to a fixed point O on the boundary of G.

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It may be added that in this theorem the slit domains could be replaced by half-plane slit domains provided G possesses at least one boundary element not equivalent to a single point.

FIGURE 2.8.

FIGURE 2.9.

Proof of the mapping theorem follows with no major difficulties from the study of the flow patterns generated by dipoles and mathematically corresponding to Abelian integrals with purely imaginary moduli of periodicity. The main step consists in cutting the domain G along critical streamlines which

FIGURE 2.10.

FIGURE 2.11.

lead from the stagnation points back to the dipole singularity. These cuts then correspond to the inner slits of the plane slit domain.

To obtain the flow patterns for such domains G of higher topological structure by experiment and measurement, a simple device is proposed. The region G may be dissected into one or more regions of genus zero which constitute G by proper boundary coordination. Each of the component regions G^i is conformally equivalent to a plane domain, again called G^i, and may be represented by, for example, an electrolytic tank. Then the boundaries of these tanks are connected mutually by a sufficiently dense net of copper wire which ensures that corresponding pairs of boundary points are "identified" in the streamline pattern. The pattern thus resulting in the system of tanks immediately furnishes the conformal mapping desired.

Figures 2.12, 2.13 and 2.14 illustrate this streamline pattern and its slit image are in the cases of a torus, a projective plane, and a Klein bottle. All these domains G are represented by a single tank with proper boundary-point identification.

FIGURE 2.12.