SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

Continuum Mechanics

GEORGE E. MASE

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OF

CONTINUUM MECHANICS

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BY

GEORGE E. MASE, Ph.D.

Professor of Mechanics
Michigan State University

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Preface

Because of its emphasis on basic concepts and fundamental principles, Continuum Mechanics has an important role in modern engineering and technology. Several undergraduate courses which utilize the continuum concept and its dependent theories in the training of engineers and scientists are well established in today's curricula and their number continues to grow. Graduate programs in Mechanics and associated areas have long recognized the value of a substantial exposure to the subject. This book has been written in an attempt to assist both undergraduate and first year graduate students in understanding the fundamental principles of continuum theory. By including a number of solved problems in each chapter of the book, it is further hoped that the student will be able to develop his skill in solving problems in both continuum theory and its related fields of application.

In the arrangement and development of the subject matter a sufficient degree of continuity is provided so that the book may be suitable as a text for an introductory course in Continuum Mechanics. Otherwise, the book should prove especially useful as a supplementary reference for a number of courses for which continuum methods provide the basic structure. Thus courses in the areas of Strength of Materials, Fluid Mechanics, Elasticity, Plasticity and Viscoelasticity relate closely to the substance of the book and may very well draw upon its contents.

Throughout most of the book the important equations and fundamental relationships are presented in both the indicial or "tensor" notation and the classical symbolic or "vector" notation. This affords the student the opportunity to compare equivalent expressions and to gain some familiarity with each notation. Only Cartesian tensors are employed in the text because it is intended as an introductory volume and since the essence of much of the theory can be achieved in this context.

The work is essentially divided into two parts. The first five chapters deal with the basic continuum theory while the final four chapters cover certain portions of specific areas of application. Following an initial chapter on the mathematics relevant to the study, the theory portion contains additional chapters on the Analysis of Stress, Deformation and Strain, Motion and Flow, and Fundamental Continuum Laws. Applications are treated in the final four chapters on Elasticity, Fluids, Plasticity and Viscoelasticity. At the end of each chapter a collection of solved problems together with several exercises for the student serve to illustrate and reinforce the ideas presented in the text.

The author acknowledges his indebtedness to many persons and wishes to express his gratitude to all for their help. Special thanks are due the following: to my colleagues, Professors W. A. Bradley, L. E. Malvern, D. H. Y. Yen, J. F. Foss and G. LaPalm each of whom read various chapters of the text and made valuable suggestions for improvement; to Professor D. J. Montgomery for his support and assistance in a great many ways; to Dr. Richard Hartung of the Lockheed Research Laboratory, Palo Alto, California, who read the preliminary version of the manuscript and gave numerous helpful suggestions; to Professor M. C. Stippes, University of Illinois, for his invaluable comments and suggestions; to Mrs. Thelma Liszewski for the care and patience she displayed in typing the manuscript; to Mr. Daniel Schaum and Mr. Nicola Monti for their continuing interest and guidance throughout the work. The author also wishes to express thanks to his wife and children for their encouragement during the writing of the book.

Michigan State University
June 1970

GEORGE E. MASE

CONTENTS

Chapter 1

Mathematical Foundations

1.1 TENSORS AND CONTINUUM MECHANICS

Continuum mechanics deals with physical quantities which are independent of any particular coordinate system that may be used to describe them. At the same time, these physical quantities are very often specified most conveniently by referring to an appropriate system of coordinates. Mathematically, such quantities are represented by *tensors.*

As a mathematical entity, a tensor has an existence independent of any coordinate system. Yet it may be specified in a particular coordinate system by a certain set of quantities, known as its *components.* Specifying the components of a tensor in one coordinate system determines the components in any other system. Indeed, the *law of transformation* of the components of a tensor is used here as a means for defining the tensor. Precise statements of the definitions of various kinds of tensors are given at the point of their introduction in the material that follows.

The physical laws of continuum mechanics are expressed by tensor equations. Because tensor transformations are linear and homogeneous, such tensor equations, if they are valid in one coordinate system, are valid in any other coordinate system. This *invariance* of tensor equations under a coordinate transformation is one of the principal reasons for the usefulness of tensor methods in continuum mechanics.

1.2 GENERAL TENSORS. CARTESIAN TENSORS. TENSOR RANK.

In dealing with general coordinate transformations between arbitrary curvilinear coordinate systems, the tensors defined are known as *general tensors.* When attention is restricted to transformations from one homogeneous coordinate system to another, the tensors involved are referred to as *Cartesian tensors.* Since much of the theory of continuum mechanics may be developed in terms of Cartesian tensors, the word "tensor" in this book means "Cartesian tensor" unless specifically stated otherwise.

Tensors may be classified by *rank,* or *order,* according to the particular form of the transformation law they obey. This same classification is also reflected in the number of components a given tensor possesses in an n-dimensional space. Thus in a three-dimensional Euclidean space such as ordinary physical space, the number of components of a tensor is 3^N, where N is the order of the tensor. Accordingly a tensor of *order zero* is specified in any coordinate system in three-dimensional space by *one* component. Tensors of order zero are called *scalars.* Physical quantities having magnitude only are represented by scalars. Tensors of *order one* have *three* coordinate components in physical space and are known as *vectors.* Quantities possessing both magnitude and direction are represented by vectors. *Second-order* tensors correspond to *dyadics.* Several important quantities in continuum mechanics are represented by tensors of rank two. Higher order tensors such as *triadics,* or tensors of order three, and *tetradics,* or tensors of order four are also defined and appear often in the mathematics of continuum mechanics.

1.3 VECTORS AND SCALARS

Certain physical quantities, such as force and velocity, which possess both magnitude and direction, may be represented in a three-dimensional space by *directed line segments* that obey the *parallelogram law of addition.* Such directed line segments are the geometrical representations of first-order tensors and are called *vectors.* Pictorially, a vector is simply an arrow pointing in the appropriate direction and having a length proportional to the magnitude of the vector. *Equal vectors* have the same direction and equal magnitudes. A *unit* vector is a vector of unit length. The *null* or *zero* vector is one having zero length and an unspecified direction. The *negative* of a vector is that vector having the same magnitude but opposite direction.

Those physical quantities, such as mass and energy, which possess magnitude only are represented by tensors of order zero which are called *scalars.*

In the *symbolic,* or *Gibbs* notation, vectors are designated by bold-faced letters such as **a**, **b**, etc. Scalars are denoted by italic letters such as a, b, λ, etc. Unit vectors are further distinguished by a caret placed over the bold-faced letter. In Fig. 1-1, arbitrary vectors **a** and **b** are shown along with the unit vector $\hat{\mathbf{e}}$ and the pair of equal vectors **c** and **d**.

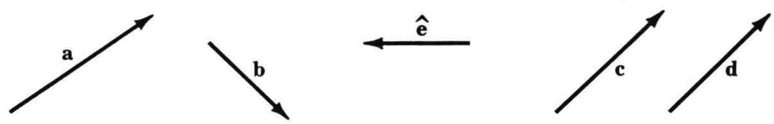


Fig. 1-1

The magnitude of an arbitrary vector **a** is written simply as a, or for emphasis it may be denoted by the vector symbol between vertical bars as $|\mathbf{a}|$.

1.4 VECTOR ADDITION. MULTIPLICATION OF A VECTOR BY A SCALAR

Vector addition obeys the *parallelogram law,* which defines the vector sum of two vectors as the diagonal of a parallelogram having the component vectors as adjacent sides. This law for vector addition is equivalent to the *triangle rule* which defines the sum of two vectors as the vector extending from the tail of the first to the head of the second when the summed vectors are adjoined head to tail. The graphical construction for the addition of **a** and **b** by the parallelogram law is shown in Fig. 1-2(*a*). Algebraically, the addition process is expressed by the vector equation

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = \mathbf{c} \qquad (1.1)$$

Vector subtraction is accomplished by addition of the negative vector as shown, for example, in Fig. 1-2(*b*) where the triangle rule is used. Thus

$$\mathbf{a} - \mathbf{b} = -\mathbf{b} + \mathbf{a} = \mathbf{d} \qquad (1.2)$$

The operations of vector addition and subtraction are commutative and associative as illustrated in Fig. 1-2(*c*), for which the appropriate equations are

$$(\mathbf{a} + \mathbf{b}) + \mathbf{g} = \mathbf{a} + (\mathbf{b} + \mathbf{g}) = \mathbf{h} \qquad (1.3)$$

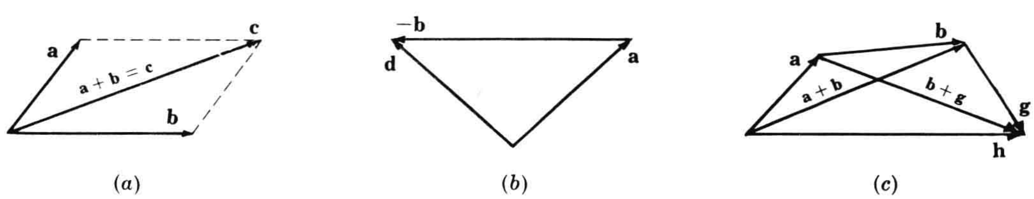


Fig. 1-2

Multiplication of a *vector* by a *scalar* produces in general a new vector having the same direction as the original but a different length. Exceptions are multiplication by zero to produce the null vector, and multiplication by unity which does not change a vector. Multiplication of the vector **b** by the scalar m results in one of the three possible cases shown in Fig. 1-3, depending upon the numerical value of m.

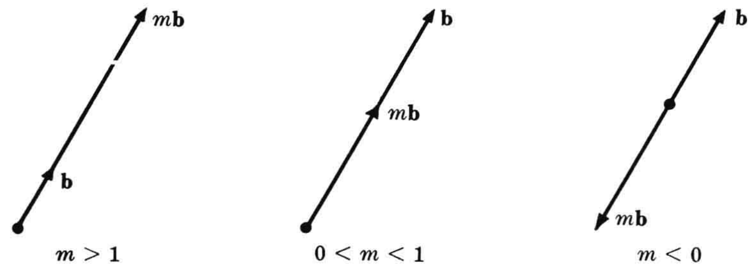


Fig. 1-3

Multiplication of a vector by a scalar is associative and distributive. Thus

$$m(n\mathbf{b}) = (mn)\mathbf{b} = n(m\mathbf{b}) \tag{1.4}$$

$$(m+n)\mathbf{b} = (n+m)\mathbf{b} = m\mathbf{b} + n\mathbf{b} \tag{1.5}$$

$$m(\mathbf{a}+\mathbf{b}) = m(\mathbf{b}+\mathbf{a}) = m\mathbf{a} + m\mathbf{b} \tag{1.6}$$

In the important case of a vector multiplied by the reciprocal of its magnitude, the result is a *unit vector* in the direction of the original vector. This relationship is expressed by the equation

$$\hat{\mathbf{b}} = \mathbf{b}/b \tag{1.7}$$

1.5 DOT AND CROSS PRODUCTS OF VECTORS

The *dot* or *scalar product* of two vectors **a** and **b** is the scalar

$$\lambda = \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = ab\cos\theta \tag{1.8}$$

in which θ is the smaller angle between the two vectors as shown in Fig. 1-4(*a*). The dot product of **a** with a unit vector $\hat{\mathbf{e}}$ gives the projection of **a** in the direction of $\hat{\mathbf{e}}$.

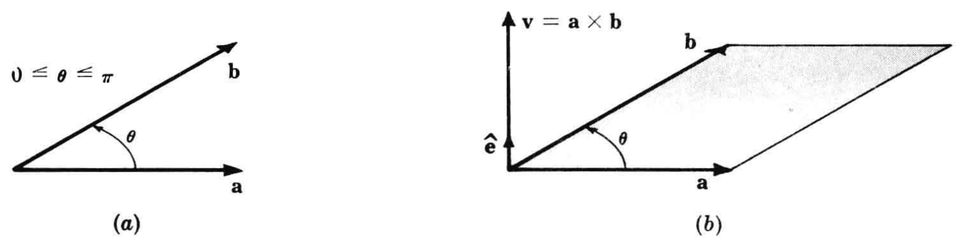


Fig. 1-4

The *cross* or *vector product* of **a** into **b** is the vector **v** given by

$$\mathbf{v} = \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = (ab\sin\theta)\,\hat{\mathbf{e}} \tag{1.9}$$

in which θ is the angle less than 180° between the vectors **a** and **b**, and $\hat{\mathbf{e}}$ is a unit vector perpendicular to their plane such that a right-handed rotation about $\hat{\mathbf{e}}$ through the angle θ carries **a** into **b**. The magnitude of **v** is equal to the area of the parallelogram having **a** and **b** as adjacent sides, shown shaded in Fig. 1-4(*b*). The cross product is not commutative.

The *scalar triple product* is a dot product of two vectors, one of which is a cross product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \lambda \tag{1.10}$$

As indicated by (*1.10*) the dot and cross operation may be interchanged in this product. Also, since the cross operation must be carried out first, the parentheses are unnecessary and may be deleted as shown. This product is sometimes written $[\mathbf{abc}]$ and called the *box product*. The magnitude λ of the scalar triple product is equal to the volume of the parallelepiped having $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as coterminous edges.

The *vector triple product* is a cross product of two vectors, one of which is itself a cross product. The following identity is frequently useful in expressing the product of $\mathbf{a}$ crossed into $\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \mathbf{w} \tag{1.11}$$

From (*1.11*), the product vector $\mathbf{w}$ is observed to lie in the plane of $\mathbf{b}$ and $\mathbf{c}$.

1.6 DYADS AND DYADICS

The *indeterminate vector product* of $\mathbf{a}$ and $\mathbf{b}$, defined by writing the vectors in juxtaposition as $\mathbf{ab}$ is called a *dyad*. The indeterminate product is not in general commutative, i.e. $\mathbf{ab} \neq \mathbf{ba}$. The first vector in a dyad is known as the *antecedent*, the second is called the *consequent*. A *dyadic* D corresponds to a tensor of order two and may always be represented as a finite sum of dyads

$$\mathsf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N \tag{1.12}$$

which is, however, never unique. In symbolic notation, dyadics are denoted by bold-faced sans-serif letters as above.

If in each dyad of (*1.12*) the antecedents and consequents are interchanged, the resulting dyadic is called the *conjugate dyadic* of D and is written

$$\mathsf{D}_c = \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + \cdots + \mathbf{b}_N\mathbf{a}_N \tag{1.13}$$

If each dyad of D in (*1.12*) is replaced by the dot product of the two vectors, the result is a scalar known as the *scalar of the dyadic* D and is written

$$\mathsf{D}_s = \mathbf{a}_1\cdot\mathbf{b}_1 + \mathbf{a}_2\cdot\mathbf{b}_2 + \cdots + \mathbf{a}_N\cdot\mathbf{b}_N \tag{1.14}$$

If each dyad of D in (*1.12*) is replaced by the cross product of the two vectors, the result is called the *vector of the dyadic* D and is written

$$\mathsf{D}_v = \mathbf{a}_1\times\mathbf{b}_1 + \mathbf{a}_2\times\mathbf{b}_2 + \cdots + \mathbf{a}_N\times\mathbf{b}_N \tag{1.15}$$

It can be shown that $\mathsf{D}_c, \mathsf{D}_s$ and D_v are independent of the representation (*1.12*).

The indeterminate vector product obeys the distributive laws

$$\mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab} + \mathbf{ac} \tag{1.16}$$

$$(\mathbf{a}+\mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc} \tag{1.17}$$

$$(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) = \mathbf{ac} + \mathbf{ad} + \mathbf{bc} + \mathbf{bd} \tag{1.18}$$

and if λ and μ are any scalars,

$$(\lambda+\mu)\mathbf{ab} = \lambda\mathbf{ab} + \mu\mathbf{ab} \tag{1.19}$$

$$(\lambda\mathbf{a})\mathbf{b} = \mathbf{a}(\lambda\mathbf{b}) = \lambda\mathbf{ab} \tag{1.20}$$

If $\mathbf{v}$ is any vector, the dot products $\mathbf{v}\cdot\mathsf{D}$ and $\mathsf{D}\cdot\mathbf{v}$ are the vectors defined respectively by

$$\mathbf{v}\cdot\mathsf{D} = (\mathbf{v}\cdot\mathbf{a}_1)\mathbf{b}_1 + (\mathbf{v}\cdot\mathbf{a}_2)\mathbf{b}_2 + \cdots + (\mathbf{v}\cdot\mathbf{a}_N)\mathbf{b}_N = \mathbf{u} \tag{1.21}$$

$$\mathsf{D}\cdot\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\cdot\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\cdot\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\cdot\mathbf{v}) = \mathbf{w} \tag{1.22}$$

In (*1.21*) D is called the *postfactor*, and in (*1.22*) it is called the *prefactor*. Two dyadics D and E are *equal* if and only if for every vector $\mathbf{v}$, either

$$\mathbf{v}\cdot\mathsf{D} = \mathbf{v}\cdot\mathsf{E} \quad \text{or} \quad \mathsf{D}\cdot\mathbf{v} = \mathsf{E}\cdot\mathbf{v} \tag{1.23}$$

The *unit dyadic*, or *idemfactor* I, is the dyadic which can be represented as

$$\mathsf{I} = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 \tag{1.24}$$

where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ constitute any orthonormal basis for three-dimensional Euclidean space (see Section 1.7). The dyadic I is characterized by the property

$$\mathsf{I}\cdot\mathbf{v} = \mathbf{v}\cdot\mathsf{I} = \mathbf{v} \tag{1.25}$$

for all vectors $\mathbf{v}$.

The cross products $\mathbf{v}\times\mathsf{D}$ and $\mathsf{D}\times\mathbf{v}$ are the dyadics defined respectively by

$$\mathbf{v}\times\mathsf{D} = (\mathbf{v}\times\mathbf{a}_1)\mathbf{b}_1 + (\mathbf{v}\times\mathbf{a}_2)\mathbf{b}_2 + \cdots + (\mathbf{v}\times\mathbf{a}_N)\mathbf{b}_N = \mathsf{F} \tag{1.26}$$

$$\mathsf{D}\times\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\times\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\times\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\times\mathbf{v}) = \mathsf{G} \tag{1.27}$$

The dot product of the dyads $\mathbf{ab}$ and $\mathbf{cd}$ is the dyad defined by

$$\mathbf{ab}\cdot\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})\mathbf{ad} \tag{1.28}$$

From (*1.28*), the dot product of any two dyadics D and E is the dyadic

$$\begin{aligned}\mathsf{D}\cdot\mathsf{E} &= (\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N)\cdot(\mathbf{c}_1\mathbf{d}_1 + \mathbf{c}_2\mathbf{d}_2 + \cdots + \mathbf{c}_N\mathbf{d}_N)\\ &= (\mathbf{b}_1\cdot\mathbf{c}_1)\mathbf{a}_1\mathbf{d}_1 + (\mathbf{b}_1\cdot\mathbf{c}_2)\mathbf{a}_1\mathbf{d}_2 + \cdots + (\mathbf{b}_N\cdot\mathbf{c}_N)\mathbf{a}_N\mathbf{d}_N = \mathsf{G}\end{aligned} \tag{1.29}$$

The dyadics D and E are said to be *reciprocal* of each other if

$$\mathsf{E}\cdot\mathsf{D} = \mathsf{D}\cdot\mathsf{E} = \mathsf{I} \tag{1.30}$$

For reciprocal dyadics, the notation $\mathsf{E} = \mathsf{D}^{-1}$ and $\mathsf{D} = \mathsf{E}^{-1}$ is often used.

Double dot and cross products are also defined for the dyads $\mathbf{ab}$ and $\mathbf{cd}$ as follows,

$$\mathbf{ab} : \mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) = \lambda, \quad \text{a scalar} \tag{1.31}$$

$$\mathbf{ab} \overset{\times}{\cdot} \mathbf{cd} = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) = \mathbf{h}, \quad \text{a vector} \tag{1.32}$$

$$\mathbf{ab} \underset{\times}{\cdot} \mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\times\mathbf{d}) = \mathbf{g}, \quad \text{a vector} \tag{1.33}$$

$$\mathbf{ab} \overset{\times}{\times} \mathbf{cd} = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\times\mathbf{d}) = \mathbf{uw}, \quad \text{a dyad} \tag{1.34}$$

From these definitions, double dot and cross products of dyadics may be readily developed. Also, some authors use the double dot product defined by

$$\mathbf{ab}\cdot\cdot\,\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = \lambda, \quad \text{a scalar} \tag{1.35}$$

A dyadic D is said to be *self-conjugate*, or *symmetric*, if

$$\mathsf{D} = \mathsf{D}_c \tag{1.36}$$

and *anti-self-conjugate*, or *anti-symmetric*, if

$$\mathsf{D} = -\mathsf{D}_c \tag{1.37}$$

Every dyadic may be expressed uniquely as the sum of a symmetric and anti-symmetric dyadic. For the arbitrary dyadic D the decomposition is

$$\mathsf{D} = \tfrac{1}{2}(\mathsf{D}+\mathsf{D}_c) + \tfrac{1}{2}(\mathsf{D}-\mathsf{D}_c) = \mathsf{G} + \mathsf{H} \tag{1.38}$$

for which $$\mathbf{G}_c = \tfrac{1}{2}(\mathbf{D}_c + (\mathbf{D}_c)_c) = \tfrac{1}{2}(\mathbf{D}_c + \mathbf{D}) = \mathbf{G} \quad \text{(symmetric)} \tag{1.39}$$

and $$\mathbf{H}_c = \tfrac{1}{2}(\mathbf{D}_c - (\mathbf{D}_c)_c) = \tfrac{1}{2}(\mathbf{D}_c - \mathbf{D}) = -\mathbf{H} \quad \text{(anti-symmetric)} \tag{1.40}$$

Uniqueness is established by assuming a second decomposition, $\mathbf{D} = \mathbf{G}^* + \mathbf{H}^*$. Then

$$\mathbf{G}^* + \mathbf{H}^* = \mathbf{G} + \mathbf{H} \tag{1.41}$$

and the conjugate of this equation is

$$\mathbf{G}^* - \mathbf{H}^* = \mathbf{G} - \mathbf{H} \tag{1.42}$$

Adding and subtracting *(1.41)* and *(1.42)* in turn yields respectively the desired equalities, $\mathbf{G}^* = \mathbf{G}$ and $\mathbf{H}^* = \mathbf{H}$.

1.7 COORDINATE SYSTEMS. BASE VECTORS. UNIT VECTOR TRIADS

A vector may be defined with respect to a particular coordinate system by specifying the *components* of the vector in that system. The choice of coordinate system is arbitrary, but in certain situations a particular choice may be advantageous. The reference system of coordinate axes provides units for measuring vector magnitudes and assigns directions in space by which the orientation of vectors may be determined.

The well-known *rectangular Cartesian coordinate system* is often represented by the mutually perpendicular axes, *Oxyz* shown in Fig. 1-5. Any vector $\mathbf{v}$ in this system may be expressed as a linear combination of three arbitrary, nonzero, noncoplanar vectors of the system, which are called *base vectors*. For base vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and suitably chosen scalar coefficients λ, μ, ν the vector $\mathbf{v}$ is given by

$$\mathbf{v} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} \tag{1.43}$$

Base vectors are by hypothesis linearly independent, i.e. the equation

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = 0 \tag{1.44}$$

is satisfied only if $\lambda = \mu = \nu = 0$. A set of base vectors in a given coordinate system is said to constitute a *basis* for that system.

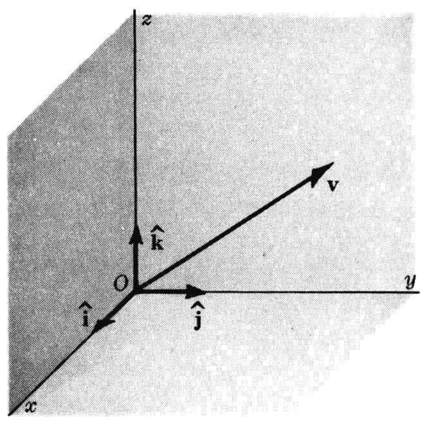


Fig. 1-5

The most frequent choice of base vectors for the rectangular Cartesian system is the set of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ along the coordinate axes as shown in Fig. 1-5. These base vectors constitute a right-handed *unit vector triad*, for which

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.45}$$

and $$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0 \tag{1.46}$$

Such a set of base vectors is often called an *orthonormal basis*.

In terms of the unit triad $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, the vector $\mathbf{v}$ shown in Fig. 1-6 below may be expressed by

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{1.47}$$

in which the Cartesian components

$$v_x = \mathbf{v}\cdot\hat{\mathbf{i}} = v\cos\alpha$$
$$v_y = \mathbf{v}\cdot\hat{\mathbf{j}} = v\cos\beta$$
$$v_z = \mathbf{v}\cdot\hat{\mathbf{k}} = v\cos\gamma$$

are the projections of $\mathbf{v}$ onto the coordinate axes. The unit vector in the direction of $\mathbf{v}$ is given according to (*1.7*) by

$$\hat{\mathbf{e}}_v = \mathbf{v}/v = (\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\hat{\mathbf{j}} + (\cos\gamma)\hat{\mathbf{k}} \qquad (1.48)$$

Since $\mathbf{v}$ is arbitrary, it follows that any unit vector will have the *direction cosines* of that vector as its *Cartesian components.*

In Cartesian component form the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a}\cdot\mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})\cdot(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) = a_xb_x + a_yb_y + a_zb_z \qquad (1.49)$$

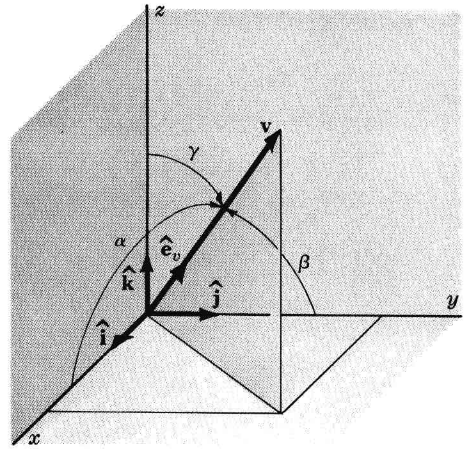


Fig. 1-6

For the same two vectors, the cross product $\mathbf{a}\times\mathbf{b}$ is

$$\mathbf{a}\times\mathbf{b} = (a_yb_z - a_zb_y)\hat{\mathbf{i}} + (a_zb_x - a_xb_z)\hat{\mathbf{j}} + (a_xb_y - a_yb_x)\hat{\mathbf{k}} \qquad (1.50)$$

This result is often presented in the determinant form

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (1.51)$$

in which the elements are treated as ordinary numbers. The triple scalar product may also be represented in component form by the determinant

$$[\mathbf{abc}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (1.52)$$

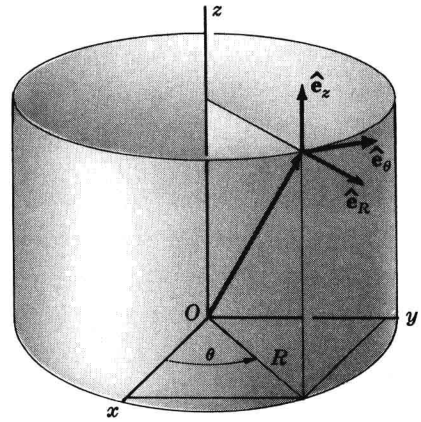
In Cartesian component form, the dyad $\mathbf{ab}$ is given by

$$\begin{aligned}\mathbf{ab} &= (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\ &= a_xb_x\hat{\mathbf{i}}\hat{\mathbf{i}} + a_xb_y\hat{\mathbf{i}}\hat{\mathbf{j}} + a_xb_z\hat{\mathbf{i}}\hat{\mathbf{k}} \\ &\quad + a_yb_x\hat{\mathbf{j}}\hat{\mathbf{i}} + a_yb_y\hat{\mathbf{j}}\hat{\mathbf{j}} + a_yb_z\hat{\mathbf{j}}\hat{\mathbf{k}} \\ &\quad + a_zb_x\hat{\mathbf{k}}\hat{\mathbf{i}} + a_zb_y\hat{\mathbf{k}}\hat{\mathbf{j}} + a_zb_z\hat{\mathbf{k}}\hat{\mathbf{k}}\end{aligned} \qquad (1.53)$$

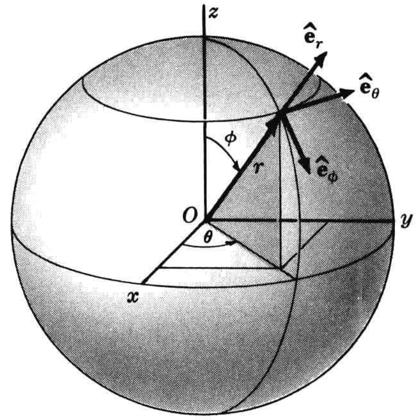
Because of the *nine* terms involved, (*1.53*) is known as the *nonion form* of the dyad $\mathbf{ab}$. It is possible to put any dyadic into nonion form. The nonion form of the idemfactor in terms of the unit triad $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ is given by

$$\mathbf{I} = \hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}} \qquad (1.54)$$

In addition to the rectangular Cartesian coordinate system already discussed, curvilinear coordinate systems such as the cylindrical (R, θ, z) and spherical (r, θ, ϕ) systems shown in Fig. 1-7 below are also widely used. Unit triads $(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ and $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ of base vectors illustrated in the figure are associated with these systems. However, the base vectors here do not all have fixed directions and are therefore, in general, functions of position.



(a) Cylindrical



(b) Spherical

Fig. 1-7

1.8 LINEAR VECTOR FUNCTIONS. DYADICS AS LINEAR VECTOR OPERATORS

A vector **a** is said to be a function of a second vector **b** if **a** is determined whenever **b** is given. This functional relationship is expressed by the equation

$$\mathbf{a} = \mathbf{f}(\mathbf{b}) \tag{1.55}$$

The function **f** is said to be linear when the conditions

$$\mathbf{f}(\mathbf{b}+\mathbf{c}) = \mathbf{f}(\mathbf{b}) + \mathbf{f}(\mathbf{c}) \tag{1.56}$$

$$\mathbf{f}(\lambda\mathbf{b}) = \lambda\mathbf{f}(\mathbf{b}) \tag{1.57}$$

are satisfied for all vectors **b** and **c**, and for any scalar λ.

Writing **b** in Cartesian component form, equation (*1.55*) becomes

$$\mathbf{a} = \mathbf{f}(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \tag{1.58}$$

which, if **f** is linear, may be written

$$\mathbf{a} = b_x\mathbf{f}(\hat{\mathbf{i}}) + b_y\mathbf{f}(\hat{\mathbf{j}}) + b_z\mathbf{f}(\hat{\mathbf{k}}) \tag{1.59}$$

In (*1.59*) let $\mathbf{f}(\hat{\mathbf{i}}) = \mathbf{u}$, $\mathbf{f}(\hat{\mathbf{j}}) = \mathbf{v}$, $\mathbf{f}(\hat{\mathbf{k}}) = \mathbf{w}$, so that now

$$\mathbf{a} = \mathbf{u}(\hat{\mathbf{i}}\cdot\mathbf{b}) + \mathbf{v}(\hat{\mathbf{j}}\cdot\mathbf{b}) + \mathbf{w}(\hat{\mathbf{k}}\cdot\mathbf{b}) = (\mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}})\cdot\mathbf{b} \tag{1.60}$$

which is recognized as a dyadic-vector dot product and may be written

$$\mathbf{a} = \mathsf{D}\cdot\mathbf{b} \tag{1.61}$$

where $\mathsf{D} = \mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}}$. This demonstrates that any linear vector function **f** may be expressed as a dyadic-vector product. In (*1.61*) the dyadic D serves as a *linear vector operator* which operates on the *argument* vector **b** to produce the *image* vector **a**.

1.9 INDICIAL NOTATION. RANGE AND SUMMATION CONVENTIONS

The components of a tensor of any order, and indeed the tensor itself, may be represented clearly and concisely by the use of the *indicial notation*. In this notation, letter indices, either subscripts or superscripts, are appended to the *generic* or *kernel* letter representing the tensor quantity of interest. Typical examples illustrating use of indices are the tensor symbols

$$a_i,\ b^j,\ T_{ij},\ F_i^{\ j},\ \epsilon_{ijk},\ R^{pq}$$

In the "mixed" form, where both subscripts and superscripts appear, the dot shows that j is the second index.

Under the rules of indicial notation, a letter index may occur either *once* or *twice* in a given term. When an index occurs unrepeated in a term, that index is understood to take on the values $1, 2, \ldots, N$ where N is a specified integer that determines the *range* of the index. Unrepeated indices are known as *free indices.* The tensorial rank of a given term is equal to the number of free indices appearing in that term. Also, correctly written tensor equations have the same letters as free indices in every term.

When an index appears *twice* in a term, that index is understood to take on all the values of its range, and the resulting terms *summed.* In this so-called *summation convention,* repeated indices are often referred to as *dummy indices,* since their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur. In general, no index occurs more than twice in a properly written term. If it is absolutely necessary to use some index more than twice to satisfactorily express a certain quantity, the summation convention must be suspended.

The number and location of the free indices reveal directly the exact tensorial character of the quantity expressed in the indicial notation. Tensors of *first order* are denoted by kernel letters bearing *one free index.* Thus the arbitrary vector **a** is represented by a symbol having a single subscript or superscript, i.e. in one or the other of the two forms,

$$a_i, \quad a^i$$

The following terms, having only one free index, are also recognized as first-order tensor quantities:

$$a_{ij}b_j, \quad F_{ikk}, \quad R^{p}_{\cdot qp}, \quad \epsilon_{ijk}u_jv_k$$

Second-order tensors are denoted by symbols having *two* free indices. Thus the arbitrary dyadic **D** will appear in one of the three possible forms

$$D^{ij}, \; D_i^{\cdot j} \quad \text{or} \quad D^{i}_{\cdot j}, \; D_{ij}$$

In the "mixed" form, the dot shows that j is the second index. Second-order tensor quantities may also appear in various forms as, for example,

$$A_{ijip}, \quad B^{ij}_{\cdot\cdot jk}, \quad \delta_{ij}u_kv_k$$

By a logical continuation of the above scheme, *third-order* tensors are expressed by symbols with *three* free indices. Also, a symbol such as λ which has no indices attached, represents a scalar, or tensor of zero order.

In ordinary physical space a *basis* is composed of three, noncoplanar vectors, and so any vector in this space is completely specified by its three components. Therefore the range on the index of a_i, which represents a vector in physical three-space, is $1, 2, 3$. Accordingly the symbol a_i is understood to represent the three components a_1, a_2, a_3. Also, a_i is sometimes interpreted to represent the ith component of the vector or indeed to represent the vector itself. For a range of three on both indices, the symbol A_{ij} represents nine components (of the second-order tensor (dyadic) **A**). The tensor A_{ij} is often presented explicitly by giving the nine components in a square array enclosed by large parentheses as

$$A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{1.62}$$

In the same way, the components of a first-order tensor (vector) in three-space may be displayed explicitly by a row or column arrangement of the form

$$a_i = (a_1, a_2, a_3) \quad \text{or} \quad a_i = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{1.63}$$

In general, for a range of N, an nth order tensor will have N^n components.

The usefulness of the indicial notation in presenting systems of equations in compact form is illustrated by the following two typical examples. For a range of three on both i and j the indicial equation

$$x_i = c_{ij}z_j \tag{1.64}$$

represents in expanded form the three equations

$$\begin{aligned} x_1 &= c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\ x_2 &= c_{21}z_1 + c_{22}z_2 + c_{23}z_3 \\ x_3 &= c_{31}z_1 + c_{32}z_2 + c_{33}z_3 \end{aligned} \tag{1.65}$$

For a range of two on i and j, the indicial equation

$$A_{ij} = B_{ip}C_{jq}D_{pq} \tag{1.66}$$

represents, in expanded form, the four equations

$$\begin{aligned} A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{12} &= B_{11}C_{21}D_{11} + B_{11}C_{22}D_{12} + B_{12}C_{21}D_{21} + B_{12}C_{22}D_{22} \\ A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\ A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22} \end{aligned} \tag{1.67}$$

For a range of three on both i and j, *(1.66)* would represent nine equations, each having nine terms on the right-hand side.

1.10 SUMMATION CONVENTION USED WITH SYMBOLIC NOTATION

The summation convention is very often employed in connection with the representation of vectors and tensors by *indexed base vectors* written in the symbolic notation. Thus if the rectangular Cartesian axes and unit base vectors of Fig. 1-5 are relabeled as shown by Fig. 1-8, the arbitrary vector $\mathbf{v}$ may be written

$$\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 \tag{1.68}$$

in which v_1, v_2, v_3 are the rectangular Cartesian components of $\mathbf{v}$. Applying the summation convention to *(1.68)*, the equation may be written in the abbreviated form

$$\mathbf{v} = v_i\hat{\mathbf{e}}_i \tag{1.69}$$

where i is a summed index. The notation here is essentially *symbolic*, but with the added feature of the *summation convention*. In such a "combination" style of notation, tensor character is not given by the *free indices rule* as it is in true indicial notation.

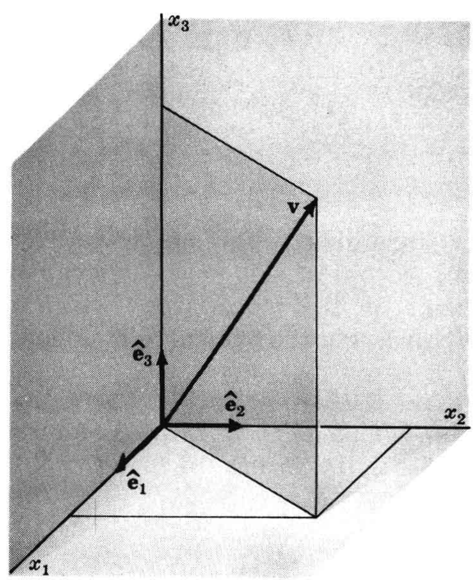


Fig. 1-8

Second-order tensors may also be represented by summation on indexed base vectors. Accordingly the dyad **ab** given in nonion form by (*1.53*) may be written

$$\mathbf{ab} = (a_i\hat{\mathbf{e}}_i)(b_j\hat{\mathbf{e}}_j) = a_i b_j \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \qquad (1.70)$$

It is essential that the sequence of the base vectors be preserved in this expression. In similar fashion, the nonion form of the arbitrary dyadic **D** may be expressed in compact notation by

$$\mathbf{D} = D_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \qquad (1.71)$$

1.11 COORDINATE TRANSFORMATIONS. GENERAL TENSORS

Let x^i represent the arbitrary system of coordinates x^1, x^2, x^3 in a three-dimensional Euclidean space, and let θ^i represent any other coordinate system $\theta^1, \theta^2, \theta^3$ in the same space. Here the numerical superscripts are labels and not exponents. Powers of x may be expressed by use of parentheses as in $(x)^2$ or $(x)^3$. The letter superscripts are indices as already noted. The *coordinate transformation equations*

$$\theta^i = \theta^i(x^1, x^2, x^3) \qquad (1.72)$$

assign to any point (x^1, x^2, x^3) in the x^i system a new set of coordinates $(\theta^1, \theta^2, \theta^3)$ in the θ^i system. The functions θ^i relating the two sets of variables (coordinates) are assumed to be single-valued, continuous, differentiable functions. The determinant

$$J = \begin{vmatrix} \dfrac{\partial\theta^1}{\partial x^1} & \dfrac{\partial\theta^1}{\partial x^2} & \dfrac{\partial\theta^1}{\partial x^3} \\ \dfrac{\partial\theta^2}{\partial x^1} & \dfrac{\partial\theta^2}{\partial x^2} & \dfrac{\partial\theta^2}{\partial x^3} \\ \dfrac{\partial\theta^3}{\partial x^1} & \dfrac{\partial\theta^3}{\partial x^2} & \dfrac{\partial\theta^3}{\partial x^3} \end{vmatrix} \qquad (1.73)$$

or, in compact form,

$$J = \left|\frac{\partial\theta^i}{\partial x^j}\right| \qquad (1.74)$$

is called the *Jacobian* of the transformation. If the Jacobian does not vanish, (*1.72*) possesses a unique inverse set of the form

$$x^i = x^i(\theta^1, \theta^2, \theta^3) \qquad (1.75)$$

The coordinate systems represented by x^i and θ^i in (*1.72*) and (*1.75*) are completely general and may be any curvilinear or Cartesian systems.

From (*1.72*), the differential vector $d\theta^i$ is given by

$$d\theta^i = \frac{\partial\theta^i}{\partial x^j}\,dx^j \qquad (1.76)$$

This equation is a prototype of the equation which defines the class of tensors known as *contravariant vectors*. In general, a set of quantities b^i associated with a point P are said to be the components of a *contravariant tensor of order one* if they transform, under a coordinate transformation, according to the equation

$$b'^i = \frac{\partial\theta^i}{\partial x^j}\,b^j \qquad (1.77)$$

where the partial derivatives are evaluated at P. In (*1.77*), b^j are the components of the tensor in the x^j coordinate system, while b'^i are the components in the θ^i system. In general