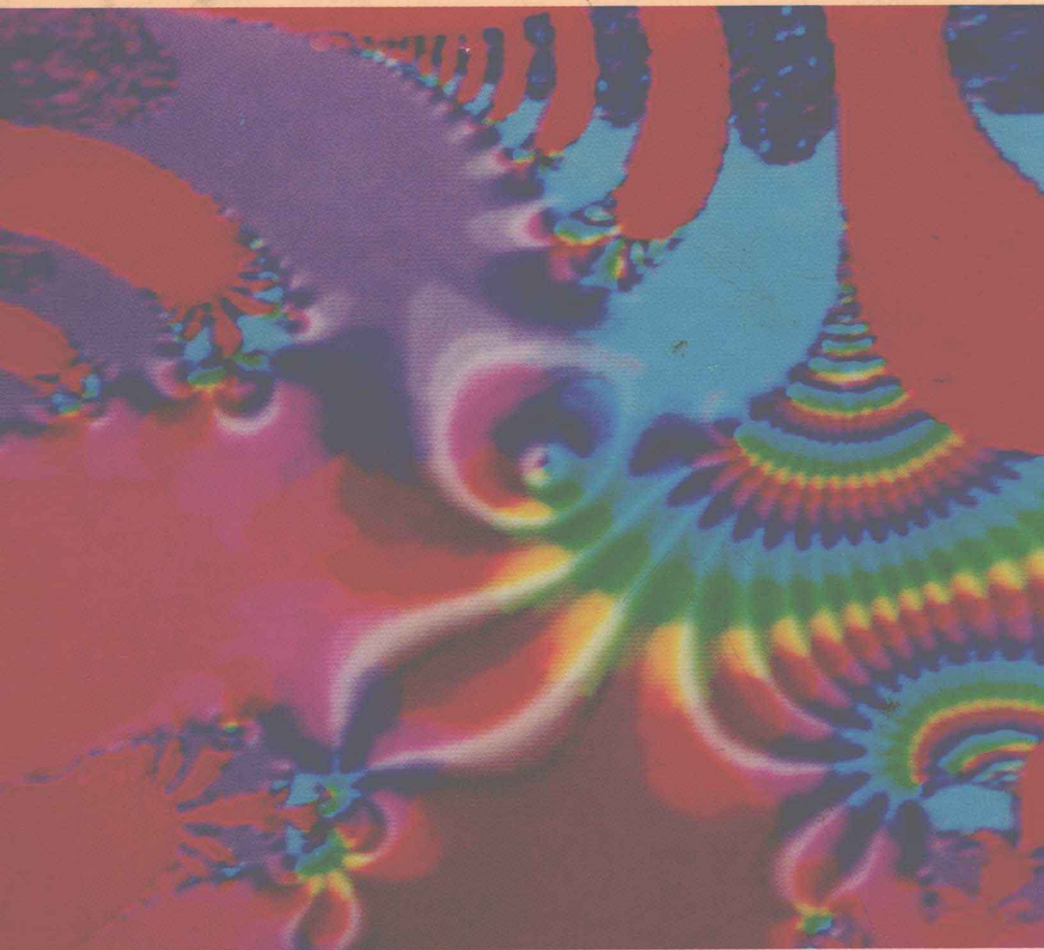
ANALYSIS OF ONE COMPLEX VARIABLE

Editor: Chung-chun Yang



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PREFACE

This volume contains a collection of papers, most of them contributed by invited speakers who presented their recent research finding at a special session entitled "Analysis of one complex variable" during the American Mathematical Society Summer Meeting held at University of Wyoming in August 1985. The main theme of the session was the contemporary studies in value-distribution theory.

A recent and significant accomplishment in this field is the verification of a more than 55-year-old conjecture raised by R. Nevanlinna - the founder of the modern value-distribution theory, which concerns the validity of the second fundamental theorem for small (deficient) functions. Almost simultaneously two different proofs appeared, one by Osgood (1985) and the other by Steinmetz (1986). In this book, Osgood gives a brief history of the methods developed to lead a final proof of the conjecture, as well as a simplification of his own proof appeared in the Journal of Number Theory in 1985. Chuang's paper reintroduces Steinmetz's argument with some technical generalizations. Gauthier's paper presents a complete proof of a result announced in an earlier paper with Arkelian: that for a certain infinite connected domain Ω, there exists a closed set of harmonic measure zero of $\partial\Omega$ which is nevertheless a set of uniqueness for $A(\Omega)$. Roy and Shah's paper proves that if two meromorphic functions each satisfying linear differential equation of second order with coefficients satisfying certain conditions, then the sum of these two functions is of bounded index and bounded value distribution. Aulaskari and Lappan's paper studies the value distribution for rotation automorphic functions and exhibits some of its applications to the theory of normal families. The material of the paper by Straus and Cayford is based on an unpublished manuscript written by late Professor Straus. It contains two theorems concerning the existence of essential singularities of functions, one of a complex variable and one of a non-Archimedian variable. Rossi's paper shows some interesting relations between the exponent convergence of the zeros of the solutions and the order of the coefficient $A(z)$ in the differential equation of the form: $W''(z) + A(z)W(z) = 0$. Gross and Yang's paper deals with the existence of fix-points of algebroidal functions. He and Yang's article presents a study on the pseudo-primeness of the product of two pseudo-prime meromorphic functions, which satisfy a certain differential equation. In Son's paper she studies the value distribution and asymptotic values of an unbounded but slow growth function. Lü and Chen's paper covers an extension of a classical result concerning Nevanlinna, Julia, and Borel directions for meromorphic functions to algebroidal functions. In a second paper of Chuang's some general inequalities involving the growth of the differential polynomial of a meromorphic function are obtained and applications of these results to differential equations and to fix-points theory are discussed. Ma's paper provides a new class of periodic entire functions of arbitrarily rapid growth which are prime functions. Dai's paper summarizes newly obtained results concerning the relationship between the order and the number of deficient values (or functions) of a meromorphic function and its derivatives. Strelitz's paper introduces a refined factor method in the Wiman-Valiron theory thereby achieving more precise results in evaluating the terms of the Taylor expansion of a transcendental entire function. Edrei presents a continuation of his studies on the Polya's conjecture: that if a meromorphic function $f(z)$ is bounded on the real axis, then every point of this axis is a limit point of the zeros of the successive derivatives of f, i.e. f', f'', f''', ... He exhibits a class of entire functions $g(z)$ such that, for each n, the zeros of the nth derivative of $g(1/z)$ are all real. Also a quantitative estimation of the number of the zeros is derived. This paper also contains many useful techniques for further research in solving the conjecture.

There are two points that need to be mentioned. Several Chinese contributors were invited and planned to attend the session, but could not come at the last moment due to the sudden change of the foreign currency policy in China. The papers by Ma, He-Yang and Gross-Yang were included though they were not scheduled for the meeting.

Finally, as an editor, I would like to thank the contributors for their support, patience and understanding given to me during the preparation of this collection. Also I want to thank Dr. Phua, Editor-in-Chief of the World Scientific Publishing Company for his endorsement and timely services.

Chung-chun Yang

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CONTENTS

VALUE DISTRIBUTION FOR ROTATION AUTOMORPHIC FUNCTIONS

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ABSTRACT

The following question is considered: if $f(z)$ is a meromorphic function in the deleted unit disk and if $g(z) = z^{\alpha}f(z)$ in the slit disk, what is the best lower bound on the area of the image of $g(z)$, considered as a subset of the Riemann sphere and not counting multiplicity. It is shown that if $\alpha = 1/2$, then the best lower bound is $\pi/2$. Some partial results are given for the general α, $0 < \alpha < 1$. In addition, it is shown that if $G(w)$ is a rotation automorphic function relative to a finitely generated Fuchsian group such that $G(w)$ omits at least one complex value, and if the area of the image of the fundamental region under G, considered as a subset of the Riemann sphere and not counting multiplicity, is less than $\pi/2$, then $G(w)$ is a normal function.

1. The Basic Question

Let $D = \{z : |z| < 1\}$, let $D^* = \{z \in D : z \neq 0\}$, and for each real number δ, let $D^*_\delta = \{z \in D^* : \delta < \arg z < \delta + 2\pi\}$. Let W denote the Riemann sphere with radius 1/2 (and thus with area π), where points

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of W are identified with points of the extended plane by stereographic projection (and thus we will consider meromorphic functions to assume values in W). We consider the following "Basic Question".

<u>Basic Question</u>. *Let* $f(z)$ *be a function meromorphic in* D^* *with an essential singularity at* $z=0$. *Let* $0<\alpha<1$, *let* δ *be a real number, and let* $g(z)=z^{\alpha}f(z)$ *for* $z\in D_{\delta}^*$. *What is the best lower bound (in terms of* α*) for the area of the set* $g(D_{\delta}^*)$ *on* W?

We are concerned here with the "best" lower bound for the set $g(D_{\delta}^*)$, where we want a bound independent of δ. It would be nice if this bound were independent of α as well, but this may be too much to expect. We note that we are discussing here the area of the *set* $g(D_{\delta}^*)$ (and thus the multiplicity of coverage is not considered).

If we decide to count multiplicity in computing the area of $g(D_{\delta}^*)$, then it can be shown that the area of $g(D_{\delta}^*)$ is infinite. To do this, we need to transfer the problem to the upper half plane. We will do this below in Theorem 5, after we have introduced the appropriate concepts. Thus, throughout this paper (except in the statement and proof of Theorem 5), we will consider the area of $g(D_{\delta}^*)$ as the area of the set $g(D_{\delta}^*)$, where multiplicity of coverage is not counted.

A few obvious remarks are in order here. If we remove the condition that $f(z)$ has an essential singularity at $z=0$ and if we suppose that $f(z)$ is meromorphic at $z=0$, then $\lim_{z\to 0} g(z)=0$ if $|f(0)|<\infty$, and $\lim_{z\to 0} g(z)=\infty$ if $f(0)=\infty$. In either case, the area of the set $g(D_{\delta}^*)$ could be very small for each δ. However, if $f(z)$ has an essential singularity at $z=0$, then we must have

$$\limsup_{\substack{z\to 0\\ z\in D_{\delta}^*}} |g(z)| = \infty \qquad \text{and} \qquad \liminf_{\substack{z\to 0\\ z\in D_{\delta}^*}} |g(z)| = 0 \tag{1}$$

for each choice of δ, by the Liouville Theorem, and the area of the set $f(D_{\delta}^*)$ must be π by Picard's Theorem.

Very little seems to be known about the Basic Question, and we know of no directly relevant results in the literature. In what follows, we will give a complete answer to the Basic Question in the special case $\alpha = 1/2$, where $\pi/2$ is the appropriate lower bound. For other values of α, we can give only fragmentary results, although we speculate that the correct lower bound is $\pi\alpha$.

In Sec. 2, we give some preliminary results and definitions. In Sec. 3, we develop our main results for the general α and also give some general results concerning rotation automorphic functions. In particular, we show in Theorem 2 that the best lower bound for the Basic Question in the case $\alpha = 1/2$ must be at least $\pi/2$. In Sec. 4, we give an example to show that this lower bound $\pi/2$ is sharp for the case $\alpha = 1/2$.

2. Some Preliminary Results

The method of attack we use involves an identification of the function $g(z)$ with a so-called "rotation automorphic" function with similar properties relative to area. To pursue this, we need some definitions.

Let $h(s)$ be a function meromorphic in a simply connected region Ω, and let Γ_Ω be a discontinuous group of conformal mappings from Ω onto itself. (The group Γ_Ω is *discontinuous* if for each $s \in \Omega$ the set $\{T(s) : T \in \Gamma_\Omega\}$ has no limit point in Ω.) We say that the function $h(s)$ is a *rotation automorphic function* (relative to the group Γ_Ω) if for each $T \in \Gamma_\Omega$ there exists a rotation S_T of the Riemann sphere such that $h(T(s)) = S_T(h(s))$ for each $s \in \Omega$. Ordinarily, the unit disk D is used as the domain Ω. In this paper, we will often use $H = \{w : \operatorname{Im} w > 0\}$ as the domain of a rotation automorphic function. Of course, D and H are conformally equivalent.

For each real number δ, let $H_\delta = \{w \in H : \delta < \operatorname{Re} w < \delta + 2\pi\}$, and let $Cl(H_\delta)$ denote the closure of H_δ. The following result gives a strong connection between the function in the Basic Question and some functions defined on H.

Theorem 1. *Let* $0 < \alpha < 1$ *and let* δ *be a real number.*

(1) *Let* $f(z)$ *be meromorphic on* D^* *and let* $g(z) = z^\alpha f(z)$ *on* D^*_δ. *Then there exists a rotation automorphic function* $G(w)$ *on* H *such that* $G(w + 2\pi) = e^{2\pi i\alpha}G(w)$ *for each* $w \in H_\delta$ *and* $g(D^*_\delta) = G(H_\delta)$ *(set equality) for each real number* δ.

(2) *Let* $G(w)$ *be a rotation automorphic function on* H *such that* $G(w + 2\pi) = e^{2\pi i\alpha}G(w)$ *for each* $w \in H$. *Then there exists a meromorphic function* $f(z)$ *in* D^* *and a function* $g(z) = z^\alpha f(z)$ *such that* $g(D^*_\delta) = G(H_\delta)$ *(set equality) for each real number* δ.

Proof. To prove (1), define $F(w) = f(e^{iw})$ for $w \in H$ and let $G(w) = e^{i\alpha w}F(w)$. It is easily verified that $G(w + 2\pi) = e^{2\pi i\alpha}G(w)$ for each $w \in H$ and that $G(w) = g(e^{iw})$. It follows that $G(H_\delta) = g(D^*_\delta)$ for each δ. This shows (1).

To show (2), first fix δ. For $z \in D^*_\delta$, define $g(z) = G(-i \log z)$ and define $f(z) = g(z)/z^\alpha$. It is easily verified that $f(z)$ is single valued and locally meromorphic on D^* (here $g(z)$ is multi-valued on D^* but locally meromorphic), so $f(z)$ is meromorphic on D^*. Now, clearly, $g(D^*_\delta) = G(H_\delta)$ (set equality) for the original fixed δ. But for $w \in H$ we have $G(w) = g(e^{iw})$, so it follows that $g(D^*_\delta) = G(H_\delta)$ for each choice of δ. This proves (2).

Two remarks are appropriate here. First, the function $G(w)$ as described in Theorem 1 (either part) is a rotation automorphic function in H relative to the group Γ generated by the transformation $T(w) = w + 2\pi$, and for this T we have $S_T(G(w)) = e^{2\pi i\alpha}G(w)$. Here, the group Γ consists of parabolic elements with the fixed point ∞. Second, if we let $H(w)$ be a rotation automorphic function such that $H(w + c) = e^{2\pi i\alpha}H(w)$ for each w H, where c is a fixed real number, then we can set $G(w) = H(cw/(2\pi))$ and get $G(w + 2\pi) = e^{2\pi i\alpha}G(w)$. Thus, the "period" is not essential in Theorem 1. In particular, part (2) of Theorem 1 could be revised to cover any rotation automorphic function for which ∞ is a fixed point of a parabolic element of the group Γ. The change from the period c to the period 2π is merely a change of scale.

We say that the function $h(s)$ meromorphic on a simply connected region Ω is a *normal function* if the family $\{h(\sigma(s)) : \sigma \in \Sigma\}$ is a normal family in the sense of Montel, where Σ denotes the collection of all conformal mappings of Ω onto itself. If $m_\Omega(s, t)$ denotes the hyperbolic distance between the points s and t in the region Ω, then $h(s)$ is a normal function on Ω if and only if

$$h^{\#}(s) = O(dm_\Omega(s)/|ds|) \quad ,$$

where $h^{\#}(s) = |h'(s)|/(1 + |h(s)|^2)$ (see [4]). In the cases where $\Omega = D$ and $\Omega = H$, the equation $h^{\#}(s) = O(dm_\Omega(s)/|ds|)$ becomes

$$(1 - |z|^2)h^{\#}(z) = O(1) \quad \text{for} \quad z \in D \quad ,$$

and

$$(\text{Im } w)h^{\#}(w) = O(1) \quad \text{for} \quad w \in H \quad , \text{ respectively.}$$

In addition, the following is known (see [2, Theorem 3, Page 415]).

Theorem A. *If* $h(s)$ *is a function meromorphic on* Ω *and* $\{s_n\}$ *is a sequence of points in* Ω *such that* $(|ds|/dm_\Omega(s))|_{s=s_n} h^{\#}(s_n) \to \infty$, *and if* $U(n, \beta) = \{s \in \Omega : m_\Omega(s, s_n) < \beta\}$, *then for each* $\beta > 0$, *the area of the set* $(\bigcup_{n \geq p} h(U(n, \beta)))$ *is* π *for each* $p > 0$.

We will use this criterion below.

3. The Main Results

We begin with the relatively simple case $\alpha = 1/2$ for the Basic Question.

Theorem 2. *Let* $\alpha = 1/2$ *in the Basic Question. Then the area of the set* $g(D_\delta^*)$ *is at least* $\pi/2$ *for each real number* δ.

Proof. Using part (1) of Theorem 1 together with its proof, if we define $G(w) = g(e^{iw})$, we have that $G(w + 2\pi) = -G(w)$ for each $w \in H$, so $G(w + 4\pi) = G(w)$, and so the function $h(z) = G(-2i \log z)$ is meromorphic for $z \in D^*$. If the area of $g(D_\delta^*)$ is less than $\pi/2$ for some $\delta > 0$, then the area of $G(H_\delta \cup H_{\delta+2\pi})$ is less than π and so the area of $h(D^*)$ is less than π. But this means that $h(z)$ is meromorphic at $z = 0$. Also, $h(z) = g(z^2) = zf(z^2)$, which means that $f(z)$ is meromorphic at $z=0$ also, in violation of the conditions in the Basic Question. It follows that the area of $g(D_\delta^*)$ is at least $\pi/2$ for each choice of δ. This proves the Theorem.

This proves that the best lower bound in the Basic Question for the case $\alpha = 1/2$ is at least $\pi/2$. We show in the next section that $\pi/2$ is actually the largest possible lower bound.

We now consider the general case, excluding the case $\alpha = 1/2$. Thus, we consider $0 < \alpha < 1$, $\alpha \neq 1/2$.

Theorem 3. *If* α, f, *and* g *are as in the Basic Question, and if* $f(z)$ *omits either of the values* 0 *or* ∞ *in* D^*, *then the area of* $g(D_\delta^*)$ *is at least* $\pi/2$ *for each real number* δ.

Proof. Form $G(w)$ as in part (1) of Theorem 1. Suppose that there exists a δ such that the area of $G(H_\delta)$ (which is the same as the area of $g(D_\delta^*)$) is less than $\pi/2$. Then $G(w)$ omits the same three values in each disk $V(s, \mu) = \{v \in H : |v-s| < \mu\}$ for $s \in H_\delta$ and $\mu < \pi$. It follows that the family $\{G_y(w) = G(w+iy) : y > 0\}$ is a normal family in H. Since $G_y(w+2\pi) = e^{2\pi i\alpha}G_y(w)$, it follows that the only constant functions possible as limits of a sequence $\{G_{y_n}(w) : y_n \to \infty\}$ on H are the functions $G_0(w) = 0$ and $G_\infty(w) = \infty$. By (1), we have

$$\limsup_{\substack{\operatorname{Im} w \to \infty \\ w \in H_\delta}} |G(w)| = \infty \qquad \text{and} \qquad \liminf_{\substack{\operatorname{Im} w \to \infty \\ w \in H_\delta}} |G(w)| = 0 \quad , \tag{2}$$

so there exists a sequence $\{y_n\}$ with $y_n \to \infty$ such that $\{G_{y_n}(w)\}$ converges uniformly on each compact subset of H to a non-constant meromorphic function $G^*(w)$. Hence, there exists a horizontal line L^* such that $G^*(w)$ is bounded away from both 0 and ∞ on $L^* \cap H_\delta$. Since $G^*(w+2\pi) = e^{2\pi i\alpha}G^*(w)$, it follows that $G^*(w)$ is bounded away from both 0 and ∞ on all of L^*. If we let $L_n = L^* + iy_n$, the vertical translation of L^* by a distance y_n, we have that there exist numbers A, B, and p such that $0 < A < |G(w)| < B < \infty$ for $w \in L_n$ and $n \geq p$. Let Q_n be the horizontal strip bounded by L_n and L_{n+1}, and consider only those n for which $n \geq p$. If $f(z)$ omits the value ∞, then $G(w)$ omits ∞ on H, so $|G(w)|$ assumes its maximum value on the set Q_n at some point in the closure of $Q_n \cap H_\delta$. Suppose there exists a point $w_0 \in Cl(Q_n \cap H_\delta)$ such that $|G(w_0)| \geq |G(w)|$ for each $w \in Q_n$. If $|G(w_0)| > B$, then w_0 is an interior point of $\{w \in Q_n : \text{Re } w_0 - 1 < \text{Re } w < \text{Re } w_0 + 1\} \cap Q_n$, in violation of the Maximum Principle. If $|G(w_0)| \leq B$, then it follows that $|G(w)| \leq B$ for all $w \in Q_n$, and this must hold for all $n \geq p$, and this violates the first part of condition (2). A similar argument involving the "Minimum Principle" will lead to a contradiction if $f(z)$ omits the value 0. Thus, we must have that the area of the set $g(D_\delta^*)$ is at least $\pi/2$ for each real number δ.

The following Lemma will be useful in dealing with rotation automorphic functions.

<u>Lemma</u>. *Let* f, g, *and* α *be as in the Basic Question, and let* $G(w)$ *be the rotation automorphic function given by part* (1) *of Theorem 1. Let* δ *be such that the area of* $G(H_\delta)$ *is less than* $\pi/2$.

(a) *If* $\alpha = 1/2$, *then* $G(w)$ *is a normal function in* H.

(b) *If* $G(w)$ *is not a normal function in* H, *then* $G(w)$ *assumes each complex value in* H.

Proof. Since the area of $G(H_\delta)$ is less than $\pi/2$, the area of $G(H_{\delta+2\pi})$ is also less than $\pi/2$. If $\alpha = 1/2$, then $G(w + 2\pi) = -G(w)$, so $G(H) = G(Cl(H_\delta)) \cup G(H_{\delta+2\pi})$, and so the area of $G(H)$ is less than π. This means that G omits three values in H, so G is a normal function in H. This proves part (a).

To prove part (b), suppose that $G(w)$ omits a value γ. By Theorem 3, $G(w)$ cannot omit either of the values 0 or ∞, so we must have $0 < |\gamma| < \infty$. Since $G(w)$ omits γ, the fact that $G(w + 2\pi) = e^{2\pi i\alpha} G(w)$ implies that $G(w)$ omits all of the values $\gamma e^{2\pi i j\alpha}$, $j = 0, \pm 1, \pm 2, \ldots$. If $\alpha = 1/2$, then $G(w)$ must be a normal function by part (a). If $\alpha \neq 1/2$, then the set $\{\gamma e^{2\pi i j\alpha}\}$ must consist of at least three values, and so $G(w)$ is a normal function if $G(w)$ omits any complex value on H. This proves part (b).