Fred S. Roberts

DISCRETE MATHEMATICAL MODELS

With applications to social, biological, and environmental problems

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To My Family

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Preface

This book is intended for a variety of audiences, including mathematicians, social scientists, biologists, environmental scientists, etc. I will argue in the Introduction that there are two directions of interaction between mathematics and any applied field. First, mathematics can be applied to that field; second, that field can stimulate the development of new mathematics. These interactions have been exhibited between mathematics and physical problems for a long time. As the interactions between mathematics and such newer areas of application as social, biological, and environmental problems become more serious, there is need to educate both mathematicians and non-mathematicians in the mathematics which is playing a role in this interaction. The by-now-common "Finite Math" books do this on an elementary level for a certain type of finite or "discrete" mathematics.[1] This book is a more advanced treatment of the discrete mathematical tools which are being used in these newer areas of application. It illustrates both the applications of mathematics to these various applied subjects and the impact of these applied subjects on the development of new mathematics.

Although this book can be used for reference, it is primarily a textbook. It can be used for a variety of courses. I have used a preliminary version of the book several times in a sophomore through senior level course at

[1]For further discussion of the nature of "discrete" mathematics, the reader is referred to the Introduction.

Rutgers University on Mathematical Models in the Social and Biological Sciences and in a junior-senior course in Graphs, Games, and their Applications. The enrollment in these courses was about 50% math majors and the rest from a variety of areas in social and biological science; it included several graduate students from disciplines outside mathematics. I have also used the book in mathematics graduate courses in Mathematical Models, in Applied Graph Theory, and in Measurement and Decisionmaking—these courses made much greater use of the proofs and theoretical material presented. The material from the book has also been used in note form in similar courses at a number of other institutions. Finally, I was fortunate enough to use the material in two Summer Institutes for college and junior college teachers of mathematics and related fields such as Operations Research, Engineering, Envrionmental Science, etc.

The 1-semester Math Models course uses material on modelling (Chapter 1), graph theory (a brief treatment[2] of Chapter 2), signed graphs and balance (Sec. 3.1), weighted digraphs and pulse processes (a brief treatment of Chapter 4), and Markov chains (a brief treatment of Chapter 5). It closes with either *n*-person games (a brief treatment of Chapter 6), group decisionmaking (a brief treatment of Chapter 7), or measurement and utility (a brief treatment of Chapter 8). In general, in the undergraduate version of this course, proofs are de-emphasized, and building and evaluation of models is emphasized. At present, the undergraduate version of this course is taught at Rutgers as a follow-up to a course in "Finite Mathematics," which covers the language of sets, elementary topics in linear algebra, counting techniques, elementary probability, etc. (The Finite Math course has some calculus as a prerequisite)

A 1-year course in Mathematical Models in the Social and Biological Sciences could cover parts of each chapter in the book. It should start with all of Chapters 1 and 2, then cover most of Chapter 3 (perhaps up through Sec. 3.5.1), and conclude with a brief treatment of the remaining chapters. Similar 1-semester or 1-year courses could emphasize environmental problems. The 1-semester graphs and games course covered most of Chapters 1, 2, 3, and 6. A 1-semester applied graph theory course covers much of Chapters 1, 2, 3, and 4. Finally, a measurement and decisionmaking course covers Chapters 1, 7, and 8, with careful treatment of exercises and supplementary readings from the references.

I have tried to keep the interdependencies in this book to a minimum, so that the book can be used in a variety of ways. The following sections and subsections in Chapter 2 are essential for Chapters 2 to 5: 2.1, 2.2.1, 2.2.4, 2.2.6, 2.3.1, 2.3.2 (not the proofs), and 2.4.1. Much of Chapter 6 can be

[2]See the table of dependencies which follows the preface for a detailed description of the "brief treatments" referred to.

read after reading Sec. 2.1 and parts of Sec. 2.2.1, and Chapters 7 and 8 have no prerequisites. Chapters 3–8 are essentially independent, though occasional reference to earlier chapters is made. If much of Chapter 3 is covered, I recommend covering all of Chapter 2 first, or beginning with the brief treatment of Chapter 2 and then returning to additional topics as they are needed. The dependencies within chapters are diagrammed at the end of the preface. Material which could be covered in a brief treatment is also described. Further guidance for what can be omitted is contained in the text.

There are a number of mathematical prerequisites for this book. The language of sets is used throughout. So are elementary logical symbols and arguments. The reader should be familiar with such terms as necessary condition, sufficient condition, converse, contrapositive, and so on. Also basic to much of the book is elementary linear algebra. However, beyond assuming certain ability to manipulate matrices and vectors (and in one place determinants), I have tried to make the book self-contained as far as linear algebra is concerned. The development also uses some probability theory, but essentially only in Chapter 5 (Markov Chains) and briefly in Sec. 8.5.2 (The Expected Utility Hypothesis). The reader who has not been exposed to the elementary theory of probability, say at the level of a book in Finite Math, will have trouble with that material. He should understand how to calculate probabilities, how to use tree diagrams, and what it means to find conditional probabilities and expected values. Counting techniques, again at the level of a Finite Math course, are also used in places. Used throughout are simple terms about functions, for example, domain, range, one-to-one, onto, etc. Some ideas from the calculus are also used in places, in particular, the idea of limit. The student without at least one semester of calculus will have trouble reading these parts.

These are the formal mathematical prerequisites for much of the book. It is my experience that the book can be used, if almost all proofs are omitted, by students with no more background than a good finite math course and a good one-semester calculus course. However, a year of calculus is strongly recommended and the student who has the added mathematical sophistication of a full course in probability or linear algebra will get much more out of this material.

Some of the subsections or proofs use more advanced mathematical tools, for example group theory and analytic or topological arguments. Other subsections simply require a fair amount of mathematical maturity. These subsections or proofs are starred or moved to the end of a section, and can be omitted without loss of continuity. Indeed, almost all proofs in this book can be omitted without loss of continuity.

The question of non-mathematical prerequisites for this book is not nearly so easy to define as the question of mathematical prerequisites. The reader of the book is not expected to be an expert in the social, biological, or environmental sciences. Indeed, the material has been used in courses populated by a great many mathematics students with little background in the applied areas discussed. Most of the basic terminology of the applied fields, when used, is explained. Indeed, it is often the purpose of the development to make precise definitions of terms from another discipline which are not defined too carefully in practice. (Examples are balance in a social structure, status in an ecological chain, etc.) Of course, any student who seriously wishes to pursue the interactions between mathematics and an applied subject had better gain some understanding of that subject as well.

This book has many exercises, usually some at the end of each section. I have long felt that the best way to learn mathematics is to do it, and so I feel that the exercises are a very important part of this book. Many of them contain additional material, not presented in the text. I have tried to arrange them in order from exercises simply asking the reader to repeat computations made in a section to more difficult theoretical ones. The hardest ones should be tractable only for the most advanced students. Some of these harder ones are marked with an asterisk *. Finally, I have added a few discussion problems and a few projects at the end of some of the sections. Some of the projects suggest a mathematical or mathematical modelling research problem which goes in a new or untried direction.

My interest in the applications of mathematics to social, biological, and environmental problems, goes back to my days as an undergraduate. This interest was nurtured along the way by many people, and in a sense they planted the seed from which this book developed. I would especially like to thank John Kemeny, Duncan Luce, Robert Norman, Dana Scott, and Patrick Suppes.

I would also like to thank the following institutions for their financial and other support of my research prior to and during the development of this book: the Institute for Advanced Study, Rutgers University, the Sloan Foundation, the RANN Program of the National Science Foundation, and the Rand Corporation, which gave me permission to use various materials I originally developed as a Rand researcher.

Prentice-Hall supplied chapter-by-chapter technical reviews which, I feel, significantly improved the quality of the book. Arthur Wester, the former Mathematics Editor at Prentice-Hall, took an early interest in the work, and his conviction that I had something different and important to say were a source of encouragement.

Many individuals supplied comments and criticisms and I cannot possibly acknowledge them all. But I would like to single out Duncan Luce and Victor Klee, who sent detailed comments on an early draft, and Kenneth Bogart, who supplied a detailed review of the next-to-last draft. William Lucas and Lloyd Shapley gave me very helpful comments about Chapter 6 (*n*-person Games). (Bill Lucas also provided me with several forums for

presentation of this material to potential users.) Jeffrey Ullman, Kenneth Bogart, David Rosen, Frank Norman, Peyton Young, and Duncan Luce provided detailed comments on other individual chapters.

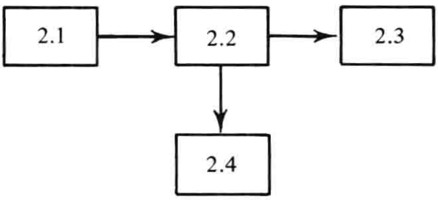
Judy Johnson and Rochelle Leibowitz found many errors in each draft, made countless useful suggestions, and carefully worked all of the exercises in the final draft. Their conscientious and enthusiastic help is hard to measure.

I alone, however, am responsible for all errors which may remain.

Finally, I would like to thank my wife Helen. As a college teacher of mathematics and statistics, and a student of mathematics and its relation to social, biological, and environmental problems, she was able to help me with many technical questions. As a wife, her patience, encouragement, understanding, and love helped me to finish this project, I hope successfully.

New Brunswick, N.J. FRED S. ROBERTS

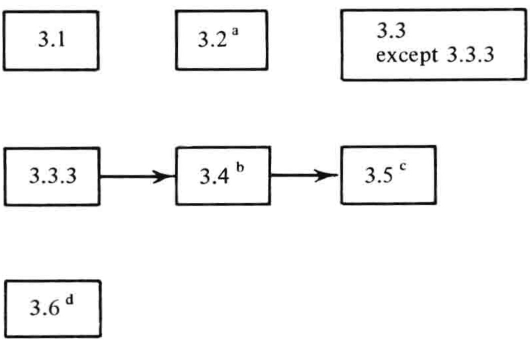
Chapter 2 Dependencies



Brief Treatment

2.1, 2.2.1, 2.2.4, 2.2.6, 2.3.1, 2.3.2, 2.4.1.

Chapter 3 Dependencies



Brief Treatment

3.3.1 and 3.3.2
or
3.1, 3.3.1, and 3.3.2
or
3.1, 3.2
or
3.3, 3.4, 3.5.1.

Notes:

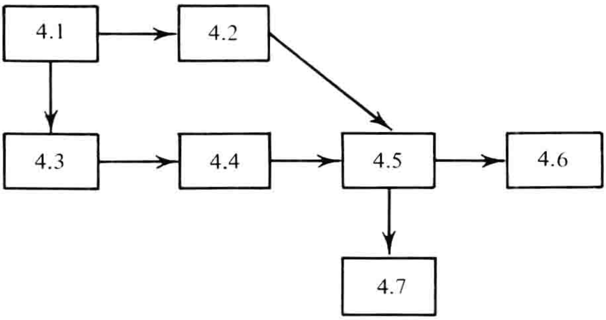
[a] Depends on Sec. 2.2.5.

[b] Secs. 3.4.3 and 3.4.4 depend in part on Theorem 3.4 of Sec. 3.2.

[c] Sec. 3.5.2 is independent of Sec. 3.5.1 and of most of the chapter.

[d] Part of Sec. 3.6.2 is dependent on Sec. 3.4.

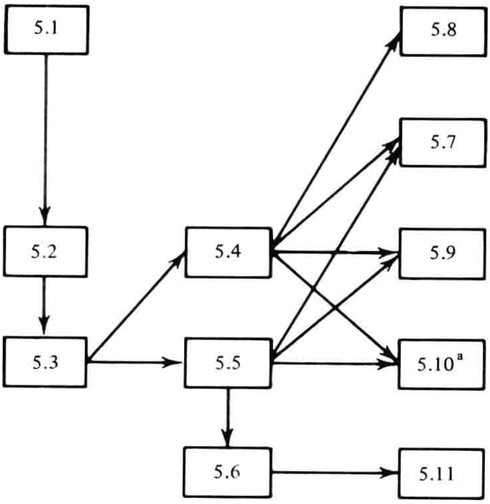
Chapter 4 Dependencies



Brief Treatment

Skip 4.7.

Chapter 5 Dependencies



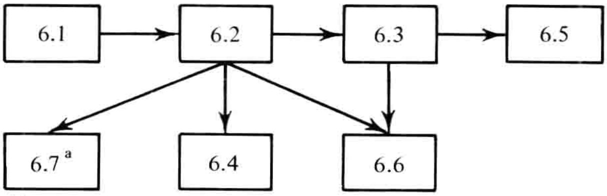
Brief Treatment

5.1, 5.2, 5.3, 5.4, 5.5, and choice of applications from 5.7, 5.8, 5.9, 5.10 and 5.11

Note:

[a] Corollary 2 to Theorem 5.12 of Sec. 5.6 is used.

Chapter 6 Dependencies



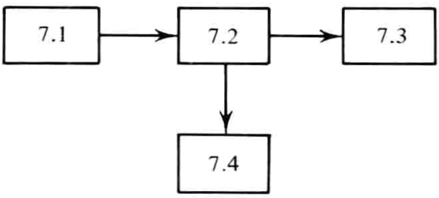
Note:

[a]Only Sec. 6.2.1 is a prerequisite

Brief Treatment

6.1, 6.2, 6.7 or
6.1, 6.2, 6.3, 6.7.

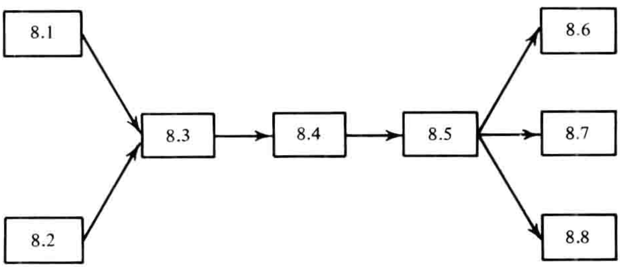
Chapter 7 Dependencies



Brief Treatment

Omit 7.3 or 7.4.

Chapter 8 Dependencies



Brief Treatment

8.1, 8.2, 8.3, 8.4, 8.5.
If there is time, add 8.6, 8.7, or 8.8.

Notation

Set Theoretical Notation

A^c	the complement of A
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
$A - B$	the difference between A and B, namely $A \cap B^c$
$A \subseteq B$	A is a subset of B
$A \subsetneqq B$	A is a proper subset of B
$A \nsubseteq B$	A is not a subset of B
$x \in B$	x is a member of B
$x \notin B$	x is not a member of B
$\varnothing$	the empty set
$\|A\|$	the cardinality of A, the number of elements in A

Logical Notation

$\sim$	not
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if (equivalence)
iff	if and only if
$\forall$	for all
$\exists$	there exists

Miscellaneous

$\mathcal{R}$	the set of real numbers
$\mathcal{R}^+$	the positive real numbers
$\mathcal{R}^n$	Euclidean n-space, i.e., the set of all n-tuples of real numbers
$[a, b]$	the closed interval consisting of all real numbers c with $a \leqq c \leqq b$.
(a, b)	the open interval consisting of all real numbers c with $a < c < b$.
$\binom{n}{k}$	the binomial coefficient $n!/k!(n-k)!$
$\equiv$	congruent to

Contents

5: Markov Chains *258*

6: *n*-Person Games *364*

7: Group Decisionmaking *425*

8: Measurement and Utility *473*

INTRODUCTION

The Scope of the Book

The interaction between mathematics and any field of application goes two ways. One way, the obvious one, is that mathematics can be applied to the other field. It can do this in a variety of ways, from solution of specific practical problems to development of broad theories. The second way, the one which is disregarded by many people, is that the applied field can be "applied to mathematics." It can do this by stimulating the development of new mathematics or by helping solve old mathematical problems.

The relations between mathematics and physics over the years clearly demonstrate this two-way interaction. Not only have many types of mathematics been used in physics, but also physical problems have been a stimulus for the development of new mathematics, for example, the calculus. Moreover, occasionally, physical models have been used to solve mathematical problems, for example certain optimization problems. (For a discussion of this point, see Polya [1954, Ch. II].)

In this book, I explore the interrelations between mathematics and applied problems from such areas as the social, biological, and environmental sciences. These interrelations are, for the most part, much newer than those between mathematics and problems from the physical sciences, and involve fields of mathematics which until recently have not often been included in mathematical education. I hope to show first that mathematics can be useful in solving problems in these newer areas of application, and second, that these areas can be a stimulus to new and interesting forms of mathematics.