Antoine Henrot

Problems for Eigenvalues of Elliptic Operators

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Preface

Problems linking the shape of a domain or the coefficients of an elliptic operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that they involve different fields of mathematics: spectral theory, partial differential equations, geometry, calculus of variations Moreover, they are very simple to state and generally hard to solve! In particular, one can find in the next pages more than 30 open problems!

In this book, we focus on extremal problems. For instance, we look for a domain which minimizes or maximizes a given eigenvalue of the Laplace operator with various boundary conditions and various geometric constraints. We also consider the case of functions of eigenvalues. We investigate similar questions for other elliptic operators, like Schrödinger, non-homogeneous membranes or composites.

The targeted audience is mainly pure and applied mathematicians, more particularly interested in partial differential equations, calculus of variations, differential geometry, spectral theory. More generally, people interested in properties of eigenvalues in other fields such as acoustics, theoretical physics, quantum mechanics, solid mechanics, could find here some answers to natural questions. For that purpose, I choose to recall basic facts and tools in the two first chapters (with only a few proofs). In chapters 3, 4 and 5, we present known results and open questions for the minimization problem of a given eigenvalue $\lambda_k(\Omega)$ of the Laplace operator with Dirichlet boundary conditions, where the unknown is here the domain Ω itself. In chapter 6, we investigate various functions of the Dirichlet eigenvalues, while chapter 7 is devoted to eigenvalues of the Laplace operator with other boundary conditions. In chapter 8, we consider the eigenvalues of Schrödinger operators: therefore, the unknown is no longer the shape of the domain but the potential V. Chapter 9 is devoted to non-homogeneous membranes and chapter 10 to more general elliptic operators in divergence form. At last, in chapter 11, we are interested in the bi-Laplace operator.

Of course no book can completely cover such a huge field of research. In making personal choices for inclusion of material, I tried to give useful complementary references, in the process certainly neglecting some relevant works. I would be grateful to hear from readers about important missing citations.

x Preface

I would like to thank Benoit Perthame who suggested in September 2004 that I write this book. Many people helped me with the enterprise, answering my questions and queries or suggesting interesting problems: Mark Ashbaugh, Friedemann Brock, Dorin Bucur, Giuseppe Buttazzo, Steve Cox, Pedro Freitas, Antonio Greco, Evans Harrell, Francois Murat, Edouard Oudet, Gerard Philippin, Michel Pierre, Marius Tucsnak. I am pleased to thank them here.

Nancy, March 2006

Antoine Henrot

Frontiers in Mathematics

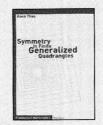
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■ Clark, J., Otago University, New Zealand / Lomp, C., Universidade di Porto, Portugal / Vanaja, N., Mumbai University, India / Wisbauer, R., Universität Düsseldorf, Germany

Lifting Modules

2006. 408 pages. Softcover ISBN 3-7643-7572-8

Extending modules are generalizations of injective modules and, dually, lifting modules generalize projective supplemented modules. There is a certain asymmetry in this duality. While the theory of extending modules is well documented in monographs and text books, the purpose of our monograph is to provide a thorough study of supplements and projectivity conditions needed to investigate classes of modules related to lifting modules. The text begins with an introduction to small submodules, the radical, variations on projectivity, and hollow dimension. The subsequent chapters consider preradicals and torsion theories (in particular related to small modules), decompositions of modules (including the exchange property and local semi-T-nilpotency), supplements in modules (with specific emphasis on semilocal endomorphism rings), finishing with a long

chapter on lifting modules, leading up their use in the theory of perfect rings, Harada rings, and quasi-Frobenius rings

Most of the material in the monograph appears in book form for the first time. The main text is augmented by a plentiful supply of exercises together with comments on further related material and on how the theory has evolved.

■ Zaharopol, R., Mathematical Reviews, Ann Arbor, USA

Invariant Probabilities of Markov-Feller Operators and Their Supports

2005. 120 pages. Softcover. ISBN 3-7643-7134-X

In this book invariant probabilities for a large class of discrete-time homogeneous Markov processes known as Feller processes are discussed. These Feller processes appear in the study of iterated function systems with probabilities, convolution operators, certain time series, etc. Rather than dealing with the processes, the transition probabilities and the operators associated with these processes are studied.

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■ De Bruyn, B., Ghent University, Ghent, Belgium

Near Polygons

2006. 276 pages. Softcover. ISBN 3-7643-7552-3

Near polygons were introduced about 25 years ago and studied intensively in the 1980s. In recent years the subject has regained interest. This monograph gives an extensive overview of the basic theory of general near polygons. The first part of the book includes a discussion of the classes of dense near polygons, regular near polygons, and glued near polygons. Also valuations, one of the most important tools for classifying dense near polygons, are treated in detail. The second part of the book discusses the classification of dense near polygons with three points per line.

The book is self-contained and almost all theorems are accompanied with proofs. Several new results are presented. Many known results occur in a more general form and the proofs are often more streamlined than their original versions. The volume is aimed at advanced graduate students and researchers in the fields of combinatorics and finite geometry.

■ Kasch, F., Universität München, Germany / Mader, A., Hawaii University

Rings, Modules, and the Total

2004. 148 pages. Softcover. ISBN 3-7643-7125-0

In a nutshell, the book deals with direct decompositions of modules and associated concepts. The central notion of "partially invertible homomorphisms", namely those that are factors of a non-zero idempotent, is introduced in a very accessible fashion. Units and regular elements are partially invertible. The "totalconsists of all elements that are not partially invertible. The total contains the radical and the singular and cosingular submodules, but while the total is closed under right and left multiplication, it may not be closed under addition. Cases are discussed where the total is additively closed. The total is particularly suited to deal with the endomorphism ring of the direct sum of modules that all have local endomorphism rings and is applied in this case. Further applications are given for torsion-free Abelian groups.

■ Krausshar, R.S., Ghent University, Ghent, Belgium

Generalized Analytic Automorphic Forms in Hypercomplex Spaces

2004. 182 pages. Softcover. ISBN 3-7643-7059-9

This book describes the basic theory of hypercomplex-analytic automorphic forms and functions for arithmetic subgroups of the Vahlen group in higher dimensional spaces.

Hypercomplex analyticity generalizes the concept of complex analyticity in the sense of considering null-solutions to higher dimensional Cauchy-Riemann type systems. Vector- and Clifford algebra-valued Eisenstein and Poincaré series are constructed within this framework and a detailed description of their analytic and number theoretical properties is provided. In particular, explicit relationships to generalized variants of the Riemann zeta function and Dirichlet L-series are established and a concept of hypercomplex multiplication of lattices is introduced.

Applications to the theory of Hilbert spaces with reproducing kernels, to partial differential equations and index theory on some conformal manifolds are also described.

■ Thas, K., Ghent University, Ghent, Belgium

Symmetry in Finite Generalized Quadrangles

2004. 240 pages. Softcover. ISBN 3-7643-6158-1

In this monograph finite generalized quadrangles are classified by symmetry, generalizing the celebrated Lenz-Barlotti classification for projective planes. The book is self-contained and serves as introduction to the combinatorial, geometrical and group-theoretical concepts that arise in the classification and in the general theory of finite generalized quadrangles, including automorphism groups, elation and translation generalized quadrangles, generalized ovals and generalized ovoids, span-symmetric generalized quadrangles, flock geometry and property (G), regularity and nets, split BN-pairs of rank 1, and the Moufang property.

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Chapter 1

Eigenvalues of elliptic operators

1.1 Notation and prerequisites

In this section, we recall the basic results of the theory of elliptic partial differential equations. The prototype of elliptic operator is the Laplacian, but the results that we state here are also valid for more general (linear) elliptic operators. For the basic facts we recall here, we refer to any textbook on partial differential equations and operator theory. For example, [36], [58], [75], [83] are good standard references.

1.1.1 Notation and Sobolev spaces

Let Ω be a bounded open set in \mathbb{R}^N . We denote by $L^2(\Omega)$ the Hilbert space of square summable functions defined on Ω and by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives (in the sense of distributions) are in $L^2(\Omega)$:

$$H^1(\Omega) := \{ u \in L^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in L^2(\Omega), \ i = 1, 2, \dots, N \}.$$

This is a Hilbert space when it is endowed with the scalar product

$$(u,v)_{H^1} := \int_{\Omega} u(x)v(x) dx + \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx$$

and the corresponding norm:

$$||u||_{H^1} := \left(\int_{\Omega} u(x)^2 dx + \int_{\Omega} |\nabla u(x)|^2 dx\right)^{1/2}.$$

In the case of Dirichlet boundary conditions, we will use the subspace $H_0^1(\Omega)$ which is defined as the closure of C^∞ functions compactly supported in Ω (functions in $C_0^\infty(\Omega)$) for the norm $\|\ \|_{H^1}$. It is also a Hilbert space. At last, $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. For some non-linear problems, for example when we are interested in the p-Laplace operator, it is more convenient to work with the spaces $L^p, p \geq 1$ instead of L^2 . In this case, the Sobolev spaces, defined exactly in the same way, are denoted by $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. These are Banach spaces.

When Ω is bounded (or bounded in one direction), we have the Poincaré inequality:

$$\exists C = C(\Omega) \text{ such that } \forall u \in H_0^1(\Omega), \ \int_{\Omega} u(x)^2 \, dx \le C \ \int_{\Omega} |\nabla u(x)|^2 \, dx \,. \tag{1.1}$$

Actually the constant C which appears in (1.1) is closely related to the eigenvalues of the Laplacian since we will see later (cf (1.36)) that the best possible constant C is nothing other than $1/\lambda_1(\Omega)$ where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

By definition, $H_0^1(\Omega)$ and $H^1(\Omega)$ are continuously embedded in $L^2(\Omega)$, but we will need later a compact embedding. This is the purpose of the following theorem.

Theorem 1.1.1 (Rellich).

- For any bounded open set Ω , the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.
- If Ω is a bounded open set with Lipschitz boundary, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Remark 1.1.2. We can weaken the assumption of Lipschitz boundary but not too much, see e.g. the book [148] for more details.

1.1.2 Partial differential equations

Elliptic operator

Let $a_{ij}(x)$, i, j = 1, ..., N be bounded functions defined on Ω and satisfying the usual ellipticity assumption:

$$\exists \alpha > 0, \text{ such that } \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \ \forall x \in \Omega$$
$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \ge \alpha |\xi|^2$$
(1.2)

where $|\xi| = (\xi_1^2 + \xi_2^2 + \dots + \xi_N^2)^{1/2}$ denotes the euclidean norm of the vector ξ . We will also assume a symmetry assumption for the a_{ij} namely:

$$\forall x \in \Omega, \forall i, j \quad a_{ij}(x) = a_{ji}(x) . \tag{1.3}$$

Let $a_0(x)$ be a bounded function defined on Ω . We introduce the linear elliptic operator L, defined on $H^1(\Omega)$ by:

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(x)u$$
 (1.4)

(derivatives are to be understood in the sense of distributions). The prototype of elliptic operator is the Laplacian:

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \tag{1.5}$$

which will be considered in the main part of this book (chapters 3 to 7). In chapter 8, we consider the Schrödinger operator $L_V u = -\Delta u + V(x)u$ where V (the potential) is a bounded function, while chapters 9 and 10 deal with more general elliptic operators. In that case, we will keep the notation L when we want to consider general operators given by (1.4). At last, in chapter 11, we consider operators of fourth order.

Remark 1.1.3. Let us remark that, since we are only interested in eigenvalue problems, we do not put any sign condition on the function $a_0(x)$ which appears in (1.4). Indeed, since $a_0(x)$ is bounded, we can always replace the operator L by $L + (\|a_0\|_{\infty} + 1)Id$, i.e. replace the function $a_0(x)$ by $a_0(x) + \|a_0\|_{\infty} + 1$ if we need a positive function in the term of order 0 of the operator L. For the eigenvalues, that would just induce a translation of $\|a_0\|_{\infty} + 1$ to the right.

Dirichlet boundary condition

Let f be a function in $L^2(\Omega)$. When we call u a solution of the Dirichlet problem

$$Lu = f \text{ in } \Omega,
 u = 0 \text{ on } \partial\Omega,$$
(1.6)

we actually mean that u is the unique solution of the variational problem

$$\begin{cases} u \in H_0^1(\Omega) \text{ and } \forall v \in H_0^1(\Omega), \\ \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x) u(x) v(x) dx = \int_{\Omega} f(x) v(x) dx . \end{cases}$$
(1.7)

Existence and uniqueness of a solution for problem (1.7) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the Poincaré inequality (1.1). Note that, according to Remark 1.1.3, we can restrict ourselves to the case $a_0(x) \geq 0$. In the sequel, we will denote by A_L^D (or $A_L^D(\Omega)$ when we want to emphasize the dependence on the domain Ω) the linear operator defined by:

$$A_L^D: L^2(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega),$$

$$f \mapsto u \text{ solution of (1.7)}.$$
(1.8)

Neumann boundary condition

In the same way, if f is a function in $L^2(\Omega)$, we will also consider u a solution of the Neumann problem

$$Lu = f \text{ in } \Omega,$$

$$\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} n_i = 0 \text{ on } \partial\Omega$$
(1.9)

(where n stands for the exterior unit normal vector to $\partial\Omega$ and n_i is its ith coordinate). For example, when $L = -\Delta$, the boundary condition reads (formally)

$$\frac{\partial u}{\partial n} = 0$$
.

It means that u is the unique solution in $H^1(\Omega)$ of the variational problem

$$\begin{cases} u \in H^{1}(\Omega) \text{ and } \forall v \in H^{1}(\Omega), \\ \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_{i}} \frac{\partial v}{\partial x_{j}} dx + \int_{\Omega} a_{0}(x)u(x)v(x) dx = \int_{\Omega} f(x)v(x) dx. \end{cases}$$
(1.10)

Existence and uniqueness of a solution for problem (1.10) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the fact that we can assume that $a_0(x) \geq 1$ (according to Remark 1.1.3). In the sequel, we will denote by A_L^N the linear operator defined by:

$$A_L^N: L^2(\Omega) \to H^1(\Omega) \subset L^2(\Omega),$$

 $f \mapsto u \text{ solution of } (1.10).$ (1.11)

Remark 1.1.4. We will also consider later, for example in chapter 7, other kinds of boundary conditions like Robin or Stekloff boundary conditions.

1.2 Eigenvalues and eigenfunctions

1.2.1 Abstract spectral theory

Let us now give the abstract theorem which provides the existence of a sequence of eigenvalues and eigenfunctions. Let H be a Hilbert space endowed with a scalar product (.,.) and recall that an operator T is a linear continuous map from H into H. We say that:

- T is positive if, $\forall x \in H, (Tx, x) > 0$,
- T is self-adjoint, if $\forall x, y \in H$, (Tx, y) = (x, Ty),
- T is compact, if the image of any bounded set is relatively compact (i.e. has a compact closure) in H.

5

Theorem 1.2.1. Let H be a separable Hilbert space of infinite dimension and T a self-adjoint, compact and positive operator. Then, there exists a sequence of real positive eigenvalues (ν_n) , $n \geq 1$ converging to 0 and a sequence of eigenvectors (x_n) , $n \geq 1$ defining a Hilbert basis of H such that $\forall n$, $T x_n = \nu_n x_n$.

Of course, this theorem can be seen as a generalization to Hilbert spaces of the classical result in finite dimension for symmetric or normal matrices (existence of real eigenvalues and of an orthonormal basis of eigenvectors).

1.2.2 Application to elliptic operators

Dirichlet boundary condition

We apply Theorem 1.2.1 to $H = L^2(\Omega)$ and the operator A_L^D defined in (1.8).

• A_L^D is positive: let $f \in L^2(\Omega)$ and $u = A_L^D f$ be the solution of (1.7). We get

$$f(f,A_L^Df)=\int_\omega f(x)u(x)\,dx=\sum_{i,j=1}^N\int_\Omega a_{ij}(x)rac{\partial u}{\partial x_i}\,rac{\partial u}{\partial x_j}\,dx+\int_\Omega a_0(x)u^2(x)\,dx\;.$$

Now, we recall that $a_0(x)$ can be taken as a positive function and then the ellipticity condition (1.2) yields the desired result. Moreover, we see that $(f, A_L^D f) > 0$ as soon as $f \neq 0$ (strict positivity).

• A_L^D is self-adjoint: let $f, g \in L^2(\Omega)$ and $u = A_L^D f, v = A_L^D g$. We have:

$$(f, A_L^D g) = \int_{\omega} f(x)v(x)dx = \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x)u(x)v(x)dx.$$

$$(1.12)$$

Now, according to the symmetry assumption (1.3) and the equation (1.7) satisfied by v, the right-hand side in (1.12) is equal to $\int_{\Omega} u(x)g(x) dx = (A_L^D f, g)$.

As a consequence of Theorem 1.2.1, there exists (u_n) a Hilbert basis of $L^2(\Omega)$ and a sequence $\nu_n \geq 0$, converging to 0, such that $A_L^D u_n = \nu_n u_n$. Actually, the ν_n are positive, since the strict positivity of A_L^D yields $\nu_n ||u_n||_{L^2} = (u_n, A_L^D u_n) > 0$.

Coming back to (1.7), we see that u_n satisfies, $\forall v \in H_0^1(\Omega)$:

$$\nu_n \left(\sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x) u_n(x) v(x) dx \right) = \int_{\Omega} u_n(x) v(x) dx$$

which means

$$L u_n = \frac{1}{\nu_n} u_n .$$

Setting $\lambda_n = \frac{1}{\nu_n}$, we have proved: