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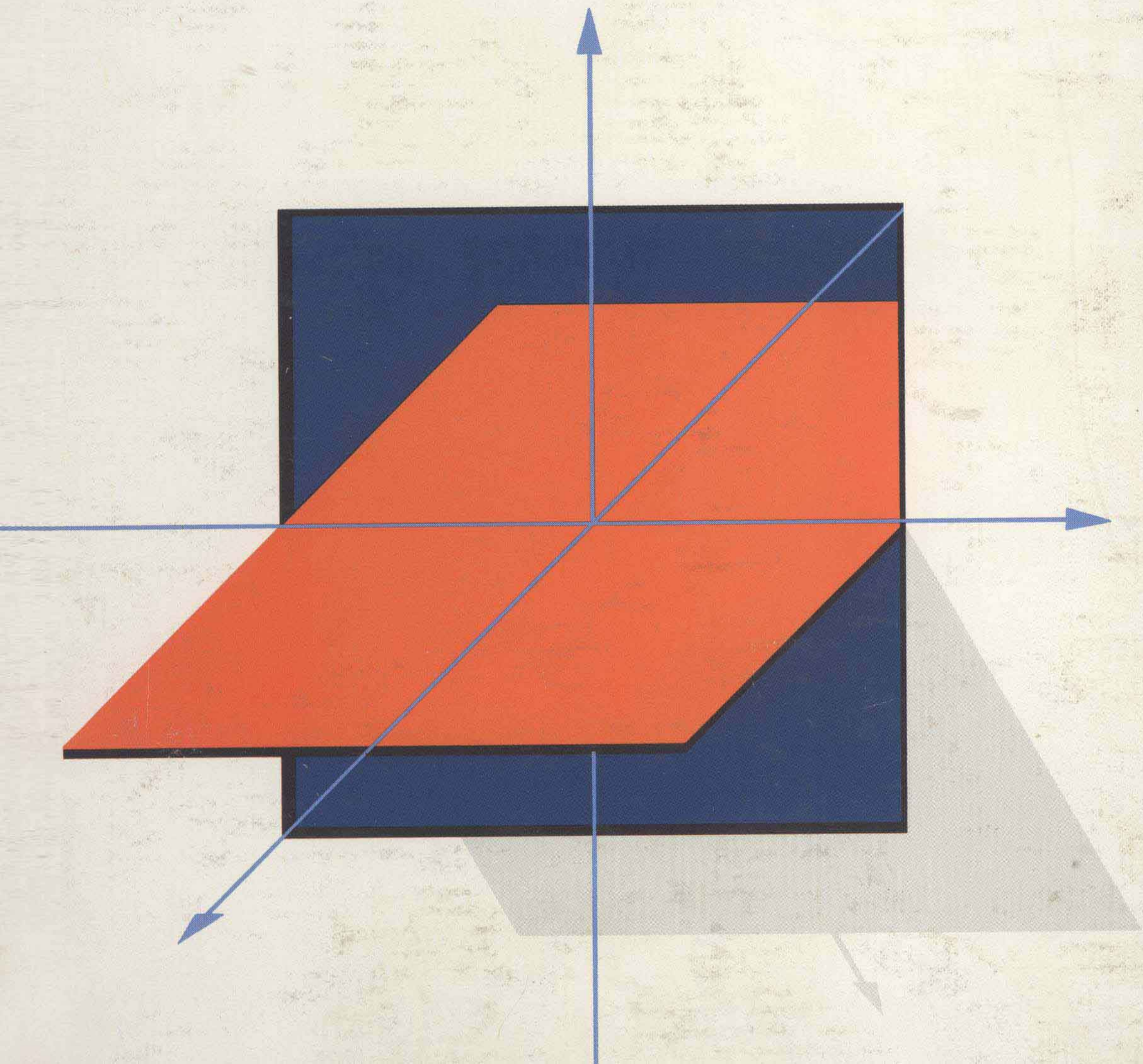
A. WAYNE ROBERTS



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THE BENJAMIN/CUMMINGS PUBLISHING COMPANY, INC.
Menlo Park, California · Reading, Massachusetts
London · Amsterdam · Don Mills, Ontario · Sydney

To Dolores

Sponsoring Editor: *S. A. Newman*
Production Editor: *Greg Hubit*
Book and Cover Designer: *John Edeen*
Artist: *Ayxa Art*

Library of Congress Cataloging in Publication Data

Roberts, A. Wayne (Arthur Wayne), 1934–
Linear algebra.

Includes index.
1. Algebra, Linear. I. Title.
QA184.R6 512′.5 81-18007
ISBN 0-8053-8302-6 AACR2

abcdefghij-HA-898765432

The Benjamin/Cummings Publishing Company, Inc.
2727 Sand Hill Road
Menlo Park, California 94025

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FOREWORD

More times than I care to count, I have taught a one-term course in elementary linear algebra, taken mainly by engineering and science students and based on one of the most popular textbooks. At first, I adhered rigorously to the text and to the official schedule for the course, with the result that the students regarded linear algebra as a sort of ritual symbol pushing, having little relevance to anything else in their experience or to their future careers. That was a shame, for while the intrinsic interest of linear algebra may be a matter of taste, there is no denying that the subject provides some of the most useful tools of applied mathematics and has many important connections with various areas of science and technology. The students, unfortunately, were in no position to appreciate the intrinsic interest of the subject, and the main part of the text provided no indication of applications or of connections with the rest of the world. The official schedule was defensible, however, for the entire term really was needed to cover basic mathematic details, thus precluding a deep discussion of any application.

What could be done to improve the situation? The "vignettes" in *Elementary Linear Algebra* provide an answer. They do not go into much detail and hence do not take much time, but they are cleverly interspersed throughout the main body of the text and thus serve to illustrate a few important connections and perhaps to convince the students of the existence of many more. I have used some of these "vignettes" as supplementary material in my own classes and have found that they greatly increased student interest in the subject.

It would be even better to have more time to cover some applications more deeply. Which areas of application? The natural candidates are linear differential equations, linear programming, and Markov chains.

In a two-term course, they could all be covered. In a one-term course, a choice must be made, depending on the instructor, on the makeup of the class, and on what material is covered in other courses. Wayne Roberts's book contains a chapter on each of these areas of application and thus provides a welcome flexibility in the use of the book.

Victor Klee
Seattle, Washington
November, 1981

PREFACE

A COMMON PROBLEM IN TEACHING LINEAR ALGEBRA

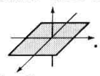
Students in a linear algebra course, perhaps more than in other mathematics courses, want to know how they will be able to use linear algebra after they have completed the course. Their instructors usually have the same nagging feeling, which they have in most courses, namely, that more should be done to relate the topic to its applications.

The solution to this problem seems obvious: Use applications as motivational material throughout the course. Those who have tried know, however, that it's not as easy as it sounds. Students don't get very far into an application before they feel that they're being told more than they really want to know, that they are being led too far afield with something they don't much care about, that this probably won't be on the test anyway, and that they are sorry they asked what the subject was good for. Also, instructors begin to worry about not getting through the standard topics because of the time devoted to applications.

A SOLUTION TO THE PROBLEM

I have tried in this book to respond to the problem in a way that avoids the pitfalls described above.

COVERAGE OF STANDARD TOPICS. The standard topics of a first course in linear algebra are covered in a straightforward manner in Chapters 1 through 6.

VIGNETTES. In every section of the first six chapters, I have included a vignette—a clearly marked short digression that relates the topic of that section to an application or to another area of mathematics. These vignettes are easily distinguishable from the body of the text by the use of a second color and by this symbol . Their location and distribution throughout the text can easily be seen by flipping through the book and looking for the swatches of second color. The titles and page numbers of these vignettes are completely listed in the table of contents. Each section (with one clearly indicated exception) is independent of its vignette. This allows instructors and students to choose the vignettes that interest them. I hope that the very placement of these vignettes in the shadows of the text will make students want to turn aside and examine them. If they do so, they should find that the vignettes have been written to give them what they want: a lighthearted, sometimes humorous indication of the kinds of things for which linear algebra is useful, without the details that would turn this fun into work. Readers of *Elementary Linear Algebra* may skip them, but I don't think they will. The vignettes, then, are quite sketchy in their treatment of the topics taken up. They are, however, just one of the responses I have made to the belief that there should be some attention paid to applications.

CHAPTERS 7, 8, AND 9. Each of the last three chapters discusses in some depth a significant application of linear algebra. Although I do not expect that anyone will be able to cover all three chapters, in a semester or two-quarter course there is time to cover at least one of them. Instructors may choose a topic because it is of personal interest, of special interest to the class, or appropriate to a particular curriculum. Taken together, these chapters offer great flexibility to the instructor, and they indicate important areas of application to any student who will peruse them. The page numbers in these chapters are shaded with gray for easy identification.

THE LEVEL OF THIS BOOK

PROOFS. The applications are in the nature of appetizers, but the text serves up the staples of a first course in linear algebra. Like all writers at this level, I have omitted a good many proofs. Where I have done this, however, I have done it in such a way as to indicate clearly what needs to be proved and why. This effort, carried consistently through the text, can be seen in the treatment of the uniqueness of the row echelon form in Chapter 1. Too often, key ideas like this one are dismissed as obvious, whereas they should be dismissed as important, difficult to prove, but easy to believe after a little experience. To this end I have included numerous exercises contrived to help students discover what they should and should not believe. Problem Set 2-4, together with the discussion in Section 2-5, illustrates both the opportunities and the warnings I have given about discovery through examples. Instructors with a taste for proofs will find, in addition to those usually included at this level, that I have included two that are often omitted: a proof that the eigenvalues of a symmetric matrix are real and a proof of the Cayley-Hamilton Theorem.

THE USE OF CALCULUS. It has become fashionable in the prefaces of linear algebra books to say that the book may be used by students who have not had calculus. That is true of this book as well, in the sense that examples and problems requiring calculus are identified with a **C** in the margin. One of the greatest attributes of linear algebra, however, is that it unifies and illuminates ideas from other areas of mathematics; this fact will be most evident to the student who has had a standard calculus course. I have not made any effort in choosing the examples or writing the vignettes to pretend that this is not so.

FOUR TYPES OF PROBLEMS

SECTION PROBLEMS. A set of problems appears at the end of each section of the book. I have attempted throughout to cover the same concepts with both even- and odd-numbered problems. I have also gone to great pains to grade these exercises and carefully link them to the text. Answers to odd-numbered problems are at the back of the book.

VIGNETTE PROBLEMS. Most section problem sets include a problem or two marked with a **V** in the margin. Such specially designated problems relate to the vignette in that section.

CALCULUS PROBLEMS. Problems that require the use of calculus are denoted by a **C**.

CHAPTER SELF-TESTS. Each chapter concludes with a self-test. This test is more than a collection of several problems from each of the sections in the chapter. Rather, it is intended to probe the students' understanding of the chapter as a whole.

SUPPLEMENTS

INSTRUCTOR'S GUIDE. The *Instructor's Guide* contains the answers to the even-numbered exercises, together with some suggestions and references that the instructor might find useful.

STUDENT SOLUTIONS MANUAL. The *Student Solutions Manual* contains solutions to selected odd-numbered exercises and to the self-tests.

ACKNOWLEDGMENTS

It was Victor Klee who first suggested to me that I write this book, and I am deeply indebted to him for his continual encouragement, for his ideas on how this course should be taught, and for many specific suggestions for vignette topics. The book would not have been written without his active involvement. I am also grateful to the many people who have reviewed this book in manuscript form, and I wish to mention in particular Donald Albers, Alan Weinstein, Alan Shuchat, and Dale Varberg. A special note of thanks goes to Bruce Edwards, who not only reviewed what I had written, but provided the basis for several vignettes dealing with computer methods in linear algebra.

I cannot ignore the contributions of students in several classes at Macalester College, and among these, Kathy Gretler and Rachael Buhse must be cited in a special way for their efforts in helping me to provide correct answers to problems. Finally, it is a great pleasure to pay tribute to the enthusiasm and determination of Susan Newman, the kind of editor every writer wishes for, but seldom finds.

A. Wayne Roberts

CONTENTS

Every section ends with a problem set, and every chapter ends with a self-test.

SYSTEMS OF LINEAR EQUATIONS 1

INTRODUCTION 1-1

Much of what we say in this chapter will not be new to our readers. Some of it may be so old as to have become rusty. It is not new, for example, to visualize the relationship between x_1 and x_2 in the equation

$$2x_1 + x_2 - 5 = 0$$

by plotting on coordinate axes some values of x_1 and x_2 that satisfy this equation (Figure 1-1). Neither is it new to find that the resulting points fall in a straight line and that the equation is consequently called **linear.**

x_1	$x_2 = -x_1 + 5$
-2	7
0	5
1	3
2	1

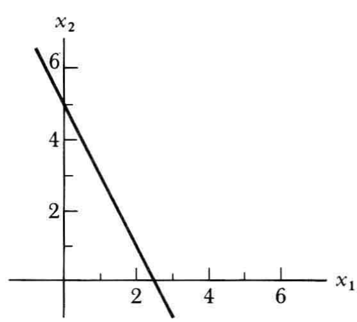


Figure 1-1

There is something, however, that will probably be new. It is the insistence on being systematic in our work. Being systematic from the beginning helps to avoid confusion later on when we deal with systems involving more than two variables. Being systematic is the only way to utilize a computer. Being systematic in our approach to specific problems provides a foundation for theoretical concepts to follow.

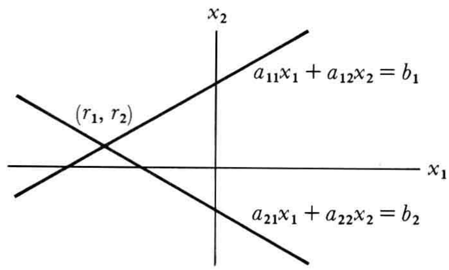
This desire to emphasize a systematic approach does present a dilemma in writing about a subject. If examples are kept simple, they invite shortcuts around the formality of the systematic methods being developed. But if examples are complicated enough to demand a systematic approach, the principles being illustrated are submerged in a sea of calculations. The way out of this dilemma is to use simple examples preceded by a plea to readers to use these examples as vehicles for learning systematic procedures. This is the plea: *Do learn the systematic methods as they are introduced.*

In this spirit of building on simple examples, let us return to the concept of a linear equation in two variables. If we can find real numbers r_1 and r_2 so that two such equations

(1)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

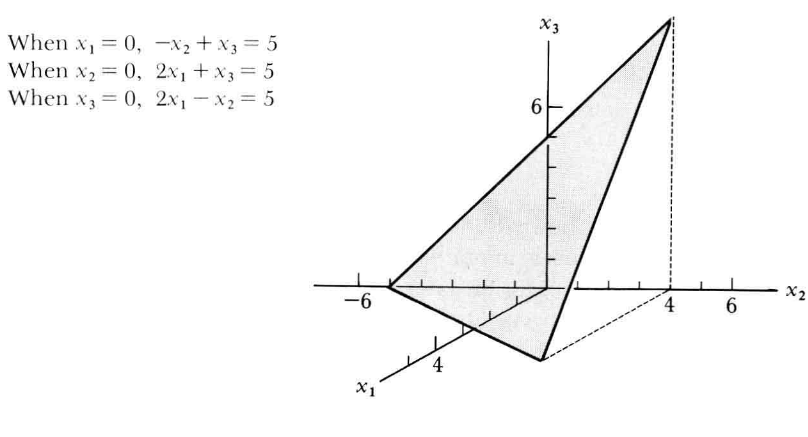
are both satisfied when we set $x_1 = r_1$ and $x_2 = r_2$, then the ordered pair (r_1, r_2) is said to be a **solution** to the system. The two straight-line graphs of these equations (Figure 1-2) intersect at the point (r_1, r_2).

Figure 1-2



Since it is obvious that two straight lines may be parallel, intersect in a single point, or be coincident, this geometric viewpoint explains why there may be no solutions, exactly one solution, or an infinite number of solutions.

Figure 1-3



The name *linear* is carried over to first degree equations in three variables such as $2x_1 - x_2 + x_3 = 5$ even though the graph, now requiring three dimensions, is a plane, not a line (Figure 1-3). In general, an equation involving n variables is called **linear** if it can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b$$

where the a_i and b are constants.

In this chapter we consider systems of m linear equations in n variables. Such a system is commonly written

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2}$$

where a_{ij} is the **coefficient** of x_j in the ith equation.

A **solution** to such a system is an ***n*-tuple** of numbers $(r_1, r_2, \ldots, r_n)$ such that if we set $x_1 = r_1, x_2 = r_2, \ldots, x_n = r_n$ in (2), all m equations will be satisfied. The set of all solutions to a system is called the **solution set** for that system. We will see that as was the case for $m = n = 2$, the solution set can always be described in one of three ways: empty, a single n-tuple, or an infinite set of n-tuples.

EQUIVALENT SYSTEMS OF EQUATIONS 1-2

TWO LINEAR EQUATIONS IN TWO VARIABLES

Much can be learned sometimes by looking carefully at something that has been familiar for a long time. To this end, let us review the solving of two linear equations in two variables as it might be explained in a beginning algebra text:

Example A

Find the solution set of

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 - 3x_2 &= -4 \end{aligned} \tag{1}$$

Our goal is to eliminate a variable from one of the equations. One way to do this is to copy the first equation, then add -2 times this copied equation to the second equation:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ - 7x_2 &= -14 \end{aligned} \tag{2}$$