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Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings

Editors

Preface

Die Regeln des Endlichen gelten im Unendlichen weiter.
(G. W. Leibniz, 1702)

Classical, real finite-dimensional differential geometry and Lie theory can be generalized in several directions. For instance, it is interesting to look for an *infinite dimensional* theory, or for a theory which works over other base fields, such as the p-adic numbers or other topological fields. The present work aims at both directions at the same time. Even more radically, we will develop the theory over certain base *rings* so that, in general, there is no dimension at all. Thus, instead of opposing "finite" and "infinite-dimensional" or "real" and "p-adic" theories, we will simply speak of "general" Lie theory or differential geometry. Certainly, this is a daring venture, and therefore I should first explain the origins and the aims of the project and say what can be done and what can *not* be done.

Preliminary versions of the present text have appeared as a series of five preprints during the years 2003 – 2005 at the Institut Elie Cartan, Nancy (still available on the web-server `http://www.iecn.u-nancy.fr/Preprint/`). The title of this work is, of course, a reference to Helgason's by now classical monograph [Hel78] from which I first learned the theory of differential geometry, Lie groups and symmetric spaces. Later I became interested in the more algebraic aspects of the theory, related to the interplay between Lie- and Jordan-theory (see [Be00]). However, rather soon I began to regret that differential geometric methods seemed to be limited to the case of the base field of real or complex numbers and to the case of "well-behaved" topological vector spaces as model spaces, such as finite-dimensional or Banach spaces (cf. the remark in [Be00, Ch.XIII] on "other base fields"). Rather surprisingly (at least for me), these limitations were entirely overcome by the joint paper [BGN04] with Helge Glöckner and Karl-Hermann Neeb where differential calculus, manifolds and Lie groups have been introduced in a very general setting, including the case of manifolds modelled on arbitrary topological vector spaces over any non-discrete topological field, and even over topological modules over certain topological base rings. In the joint paper [BeNe05] with Karl-Hermann Neeb, a good deal of the results from [Be00] could be generalized to this very general framework, leading, for example, to a rich supply of infinite-dimensional symmetric spaces.

As to differential geometry on infinite dimensional manifolds, I used to have the impression that its special flavor is due to its, sometimes rather sophisticated, functional analytic methods. On the other hand, it seemed obvious that the "purely differential" aspects of differential geometry are algebraic in nature and thus should be understandable without all the functional analytic flesh around it. In other words, one should be able to formalize the fundamental differentiation process such that its general algebraic structure becomes visible. This is indeed possible, and one possibility is opened by the general differential calculus mentioned above: for me, the most striking result of this calculus is the one presented in Chapter 6 of this work (Theorem 6.2), saying that the usual "tangent functor" T of differential geometry can, in this context, be interpreted as a functor of scalar extension from the base field or ring $\mathbb{K}$ to the *tangent ring* $T\mathbb{K} = \mathbb{K} \oplus \varepsilon\mathbb{K}$ with relation $\varepsilon^2 = 0$, sometimes also called the *dual numbers over* $\mathbb{K}$. Even though the result itself may not seem surprising (for instance, in algebraic geometry it is currently used), it seems that the proof in our context is the most natural one that one may imagine. In some sense, all the rest of the text can be seen as a try to exploit the consequences of this theorem.

I would like to add a remark on terminology. Classical "calculus" has two faces, namely *differential calculus* and *integral calculus*. In spite of its name, classical "differential geometry" also has these two faces. However, in order to avoid misunderstandings, I propose to clearly distinguish between "differential" and "integral" geometry, the former being the topic of the present work, the latter being restricted to special contexts where various kinds of integration theories can be implemented (cf. Appendix L). In a nutshell, differential geometry in this sense is the theory of k-th order Taylor expansions, for any $k \in \mathbb{N}$, of objects such as manifolds, vector bundles or Lie groups, without requiring convergence of the Taylor series. In the French literature, this approach is named by the somewhat better adapted term "développement limité" (cf., e.g., [Ca71]), which could be translated by "limited expansion". Certainly, to a mathematician used to differential geometry in real life, such a rigorous distinction between "differential" and "integral" geometry may seem pedantic and purely formal – probably, most readers would prefer to call "formal differential geometry" what remains when integration is forbidden. However, the only "formal" construction that we use is the one of the tangent bundle, and hence the choice of terminology depends on wether one considers the tangent bundle as a genuine object of differential geometry or not. Only in the very last chapter (Chapter 32) we pass the frontier separating true manifolds from formal objects by taking the projective limit $k \to \infty$ of our k-th order Taylor expansions, thus making the link with approaches by formal power series.

The present text is completely self-contained, but by no means it is an exhaustive treatise on the topics mentioned in the title. Rather, I wanted to present a first approximation of a new theory that, in my opinion, is well-adapted to capture some very general aspects of differential calculus and differential geometry and thus might be interesting for mathematicians and physicists working on topics that are related to these aspects.

Acknowledgements. I would not have dared to tackle this work if I had not had teachers who, at a very early stage, attracted my interest to foundational questions of calculus. In particular, although I do not follow the lines of thought of non-standard analysis, I am indebted to Detlef Laugwitz from whose reflections on the foundations of analysis I learned a lot. (For the adherents of non-standard analysis, I add the remark that the field $^*\mathbb{R}$ of non-standard numbers is of course admitted as a possible base field of our theory.)

I would also like to thank Harald Löwe for his careful proofreading of the first part of the text, and all future readers for indicating to me errors that, possibly, have remained.

Wolfgang Bertram Nancy, february 2005.

Abstract. The aim of this work is to lay the foundations of differential geometry and Lie theory over the general class of topological base fields and -rings for which a differential calculus has been developed in [BGN04], without any restriction on the dimension or on the characteristic. Two basic features distinguish our approach from the classical real (finite or infinite dimensional) theory, namely the interpretation of tangent- and jet functors as functors of *scalar extensions* and the introduction of *multilinear bundles* and *multilinear connections* which generalize the concept of vector bundles and linear connections.

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Contents

0. Introduction

The classical setting of differential geometry is the framework of finite-dimensional real manifolds. Later, parts of the theory have been generalized to various kinds of infinite dimensional manifolds (cf. [La99]), to analytic manifolds over ultrametric fields (finite-dimensional or Banach, cf. [Bou67] or [Se65]) or to more formal concepts going beyond the notion of manifolds such as "synthetic differential geometry" and its models, the so-called "smooth toposes" (cf. [Ko81] or [MR91]). The purpose of the present work is to unify and generalize some aspects of differential geometry in the context of manifolds over very general base fields and even -rings, featuring fundamental structures which presumably will persist even in further generalizations beyond the manifold context. We have divided the text into seven main parts of which we now give a more detailed description.

I. Basic Notions

0.1. *Differential calculus.* Differential geometry may be seen as the "invariant theory of differential calculus", and *differential calculus* deals, in one way or another, with the "infinitesimally small", either in the language of limits or in the language of infinitesimals – in the words of G.W. Leibniz: "Die Regeln des Endlichen gelten im Unendlichen weiter"[1] (quoted from [Lau86, p. 88]). In order to formalize this idea, take a topological commutative unital ring $\mathbb{K}$ (such as $\mathbb{K} = \mathbb{R}$ or any other model of the continuum that you prefer) and let "das Endliche" ("the finite") be represented by the invertible scalars $t \in \mathbb{K}^\times$ and "das unendlich (Kleine)" ("the infinitely (small)") by the non-invertible scalars. Then "Leibniz principle" may be interpreted by the assumption that $\mathbb{K}^\times$ is dense in $\mathbb{K}$: the behaviour of a continuous mapping at the infinitely small is determined by its behavior on the "finite universe". For instance, if $f : V \supset U \to W$ is a map defined on an open subset of a topological $\mathbb{K}$-module, then for all invertible scalars t in some neighborhood of 0, the "slope" (at $x \in U$ in direction $v \in V$ with parameter t)

$$f^{[1]}(x, v, t) := \frac{f(x + tv) - f(x)}{t} \tag{0.1}$$

is defined. We say that f is *of class* C^1 *(on* U*)* if the slope extends to a *continuous* map $f^{[1]} : U^{[1]} \to W$, where $U^{[1]} = \{(x, v, t) | \, x + tv \in U\}$. By Leibniz' principle, this extension is unique if it exists. Put another way, the fundamental relation

$$f(x + tv) - f(x) = t \cdot f^{[1]}(x, v, t) \tag{0.2}$$

continues to hold for all $(x, v, t) \in U^{[1]}$. The *derivative of* f *at* x *(in direction* v*)* is the continuation of the slope to the most singular of all infinitesimal elements, namely to $t = 0$:

$$df(x)v := f^{[1]}(x, v, 0). \tag{0.3}$$

[1] "The rules of the finite continue to hold in the infinite."

Now all the basic rules of differential calculus are easily proved, from the Chain Rule via Schwarz' Lemma up to Taylor's formula (see Chapter 1 and [BGN04]) – the reader who compares the proofs with the "classical" ones will find that our approach at the same time simplifies and generalizes this part of calculus.

0.2. *Differential geometry versus integral geometry.* The other half of Newton's and Leibniz' "calculus" is *integral* calculus. For the time being, the simple equation (0.2) has no integral analog – in other words, we cannot reverse the process of differentiation over general topological fields or rings (not even over the real numbers if we go too far beyond the Banach-space set-up). Thus in general we will not even be able to determine the space of all solutions of a "trivial" differential equation such as $df = 0$, and therefore no integration will appear in our approach to differential geometry: there will be no existence theorems for flows, for geodesics, or for parallel transports, no exponential map, no Poincaré lemma, no Frobenius theorem... what remains is really *differential* geometry, in contrast to the mixture of integration and differentiation that we are used to from the finite-dimensional real theory. At a first glance, the remaining structure may look very poor; but we will find that it is in fact still tremendously complicated and rich. In fact, we will see that many classical notions and results that usually are formulated on the global or local level or at least on the level of germs, continue to make sense on the level of *jets (of arbitrary order)*, and although we have in general no notion of convergence, we can take some sort of a limit of these objects, namely a *projective limit.*

II. Interpretation of tangent objects via scalar extensions

0.3. *Justification of infinitesimals.* We define *manifolds* and *tangent bundles* in the classical way using charts and atlasses (Ch. 2 and 3); then, intuitively, we may think of the tangent space T_pM of M at p as a "(first order) infinitesimal neighborhood" of p. This idea may be formalized by writing, with respect to some fixed chart of M, a tangent vector at the point p in the form $p + \varepsilon v$, where ε is a "very small" quantity (say, Planck's constant), thus expressing that the tangent vector is "infinitesimally small" compared to elements of M (in a chart). The property of being "very small" can mathematically be expressed by requiring that ε^2 is zero, compared with all "space quantities". This suggests that, if M is a manifold modelled on the $\mathbb{K}$-module V, then the tangent bundle should be modelled on the space $V \oplus \varepsilon V := V \times V$, which is considered as a module over the ring $\mathbb{K}[\varepsilon] = \mathbb{K} \oplus \varepsilon\mathbb{K}$ of *dual numbers over* $\mathbb{K}$ (it is constructed from $\mathbb{K}$ in a similar way as the complex numbers are constructed from $\mathbb{R}$, replacing the condition $i^2 = -1$ by $\varepsilon^2 = 0$). All this would be really meaningful if TM were a manifold not only over $\mathbb{K}$, but also over the extended ring $\mathbb{K}[\varepsilon]$. In Chapter 6 we prove that this is indeed true: the "tangent functor" T can be seen as a functor of scalar extension by dual numbers (Theorem 6.2). Hence one may use the "dual number unit" ε when dealing with tangent bundles with the same right as the imaginary unit i when dealing with complex manifolds. The proof of Theorem 6.2 is conceptual and allows to understand why dual numbers naturally appear in this context: the basic idea is that one can "differentiate the definition of differentiability" – we write (0.2) as a commutative diagram, apply the tangent functor to this diagram and get a diagram of the same form, but taken over the tangent ring $T\mathbb{K}$ of $\mathbb{K}$. The upshot is now that $T\mathbb{K}$ is nothing but $\mathbb{K}[\varepsilon]$.

Theorem 6.2 may be seen as a bridge from differential geometry to algebraic geometry where the use of dual numbers for modelizing tangent objects is a standard technique (cf. [GD71], p. 11), and also to classical geometry (see article of Veldkamp in [Bue95]). It is all the more surprising that in most textbooks on differential geometry no trace of the dual number unit ε can be found – although it clearly leaves a trace which is well visible already in the usual real framework: in the same way as a complex structure induces an almost complex structure on the underlying real manifold, a dual number structure induces a tensor field of endomorphisms having the property that its square is zero; let us call this an "almost dual structure". Now, there is a *canonical* almost dual structure on every tangent bundle (the almost trivial proof of this is given in Section 4.6). This canonical almost dual structure is "integrable" in the sense that its kernel is a distribution of subspaces that admits integral submanifolds – namely the fibers of the bundle TM. Therefore Theorem 6.2 may be seen as an analog of the well-known theorem of Newlander and Nirenberg: our integrable almost dual structure induces on TM the structure of a manifold over the ring $\mathbb{K}[\varepsilon]$.

Some readers may wish to avoid the use of rings which are not fields, or even to stay in the context of the real base field. In principle, all our results that do not directly involve dual numbers can be proved in a "purely real" way, i.e., by interpreting ε just as a formal (and very useful !) label which helps to distinguish two copies of V which play different rôles (fiber and base). But in the end, just as the "imaginary unit" i got its well-deserved place in mathematics, so will the "infinitesimal unit" ε.

0.4. *Further scalar extensions.* The suggestion to use dual numbers in differential geometry is not new – one of the earliest steps in this direction was by A. Weil ([W53]); one of the most recent is [Gio03]. However, most of the proposed constructions are so complicated that one is discouraged to iterate them. But this is exactly what makes the dual number formalism so useful: for instance, if TM is the scalar extension of M by $\mathbb{K}[\varepsilon_1]$, then the double tangent bundle $T^2M := T(TM)$ is simply the scalar extension of M by the ring

$$TTK := T(T\mathbb{K}) = \mathbb{K}[\varepsilon_1][\varepsilon_2] \cong \mathbb{K} \oplus \varepsilon_1\mathbb{K} \oplus \varepsilon_2\mathbb{K} \oplus \varepsilon_1\varepsilon_2\mathbb{K}, \tag{0.4}$$

and so on for all higher tangent bundles T^kM. As a matter of fact, most of the important notions of differential geometry deal, in one way or another, with the second order tangent bundle T^2M (e.g. Lie bracket, exterior derivative, connections) or with T^3M (e.g. curvature) or even higher order tangent bundles (e.g. covariant derivative of curvature). Therefore second and third order differential geometry really is the central part of all differential geometry, and finding a good notation concerning second and third order tangent bundles becomes a necessity. Most textbooks, if at all TTM is considered, use a component notation in order to describe objects related to this bundle. In this situation, the use of *different* symbols $\varepsilon_1, \varepsilon_2, \ldots$ for the infinitesimal units of the various scalar extensions is a great notational progress, combining algebraic rigour and transparency. It becomes clear that many structural features of T^kM are simple consequences of corresponding structural features of the rings $T^k\mathbb{K} = \mathbb{K}[\varepsilon_1, \ldots, \varepsilon_k]$ (cf. Chapter 7). For instance, on both objects there is a canonical action of the permutation group Σ_k; the subring $J^k\mathbb{K}$ of $T^k\mathbb{K}$ fixed under this action is called a *jet ring* (Chapter 8) – in the case of characteristic zero this is a truncated polynomial ring $\mathbb{K}[x]/(x^{k+1})$. Therefore the subbundle J^kM of T^kM fixed under the action of Σ_k can be seen as the scalar extension of M by the jet ring $J^k\mathbb{K}$; it can be interpreted as the *bundle of k-jets of (curves in) M*. The bundles T^kM and J^kM are the stage on which higher order differential geometry is played.

III. Second order differential geometry

0.5. *Bilinear bundles and connections.* The theory of *linear connections* and their *curvature* is a chief object of general differential geometry in our sense. When studying higher tangent bundles $T^2M, T^3M, \ldots$ or tangent bundles $TF, T^2F, \ldots$ of linear (i.e. vector-) bundles $p : F \to M$, one immediately encounters the problem that these bundles are *not linear bundles over M*. One may ask, naively: "if it is not a linear bundle, so what is it then ?" The answer can be found, implicitly, in all textbooks where the structure of TTM (or of TF) is discussed in some detail (e.g. [Bes78], [La99], [MR91]), but it seems that so far no attempt has been made to separate clearly linear algebra from differential calculus. Doing this, one is lead to a purely algebraic concept which we call *bilinear space* (cf. Appendix Multilinear Geometry). A *bilinear bundle* is simply a bundle whose fibers carry a structure of bilinear space which is preserved under change of bundle charts (Chapter 9). For instance, TTM and TF are bilinear bundles. Basically, a bilinear space E is given by a whole collection of linear (i.e., $\mathbb{K}$-module-) structures, parametrized by some other space E', satisfying some purely algebraic axioms. We then say that these linear structures are *bilinearly related.* A *linear connection* is simply given by singling out, in a fiberwise smooth way, in each fiber *one* linear structure among all the bilinearly related linear structures.

Of course, in the classical case our definition of a linear connection coincides with the usual ones – in fact, in the literature one finds many different definitions of a connection, and, being aware of the danger that we would be exposed to[1], it was not our aim to add a new item to this long list. Rather, we hope to have found the central item around which one can organize the spider's web of related notions such as *sprays, connectors, connection one-forms* and *covariant derivatives* (Chapters 10, 11, 12). We resisted the temptation to start with general Ehresmann connections on general fiber bundles. In our approach, principal bundles play a much less important rôle than in the usual framework – a simple reason for this is that the general (continuous) linear automorphism group $\mathrm{Gl}_{\mathbb{K}}(V)$ of a topological $\mathbb{K}$-module is in general no longer a Lie group, and hence the prime example of a principal bundle, the frame bundle, is not at our disposition.

[1] [Sp79, p. 605]: "I personally feel that the next person to propose a new definition of a connection should be summarily executed."

IV. Third and higher order differential geometry

0.6. *Multilinear bundles and multilinear connections.* The step from second order to third order geometry is important and conceptually not easy since it is at this stage that *non-commutativity* enters the picture: just as TTM is a bilinear bundle, the higher order tangent bundles $T^k M$ carry the structure of a *multilinear bundle* over M. The fibers are *multilinear spaces* (Chapter MA), which again are defined to be a set E with a whole collection of linear structures (called *multilinearly related*), parametrized by some space E', and a *multilinear connection* is given by fixing *one* of these multilinearly related structures (Chapters 15 and 16). In case $k = 2$, E' is an *abelian* (in fact, vector) group, and therefore the space of linear connections is an affine space over $\mathbb{K}$. For $k > 2$, the space E' is a *non-abelian* group, called the *special multilinear group* $\mathrm{Gm}^{1,k}(E)$.

0.7. *Curvature.* Our strategy of analyzing the higher order tangent bundle $T^k M$ is by trying to define, in a most canonical way, a sequence of multilinear connections on $TTM, T^3M, \ldots$, starting with a linear connection in the sense explained above. In other words, we want to define vector bundle structures over M on $TTM, T^3M, \ldots$, in such a way that they define, by restriction, also vector bundle structures over M on the jet bundles $J^1M, J^2M, \ldots$ which are compatible with taking the projective limit $J^\infty M$. Thus on the filtered, non-linear bundles $T^k M$ and $J^k M$ we wish to define a sequence of graded and linear structures.

It turns out that there is a rather canonical procedure how to define such a sequence of linear structures: we start with a linear connection L and define suitable derivatives $DL, D^2L, \ldots$ (Chapter 17). Unfortunately, since second covariant derivatives do in general not commute, the multilinear connection $D^k L$ on $T^{k+1}M$ is in general not invariant under the canonical action of the permutation group Σ_{k+1}. Thus, in the worst case, we get $(k+1)!$ different multilinear connections in this way. Remarkably enough, they all can be restricted to the jet bundle $J^{k+1}M$ (Theorem 19.2), but in general define still $k!$ different linear structures there. Thus, for $k = 2$, we get 2 linear structures on TTM: their difference is the *torsion* of L. For $k = 3$, we get 6 linear structures on T^3M, but their number reduces to 2 if L was already chosen to be torsionfree. In this case, their "difference" is a tensor field of type $(2, 1)$, agreeing with the classical *curvature tensor* of L (Theorem 18.3). The induced linear structure on J^3M is then unique, which is reflected by *Bianchi's identity* (Chapter 18 and Section SA.10). For general $k \geq 3$, we get a whole bunch of *curvature operators*, whose combinatorial structure seems to be an interesting topic for further work.

If L is flat (i.e, it has vanishing torsion and curvature tensors), then the sequence $DL, D^2L, \ldots$ is indeed invariant under permutations and under the so-called *shift operators* (Chapter 20), which are necessary conditions for integrating them to a "canonical chart" associated to a connection (Theorem 20.7). For general connections, the problem of defining a permutation and shift-invariant sequence of multilinear connections is far more difficult (see Section 0.16, below).

0.8. *Differential operators, duality and "representation theory."* A complete theory of differential geometry should not exclude the cotangent-"functor" T^*, which in some sense plays a more important rôle in most modern theories than the tangent functor T. In fact, the duality between T and T^* corresponds to a duality between *curves* $\gamma : \mathbb{K} \to M$ and *(scalar) functions* $f : M \to \mathbb{K}$. If the theory is rather based on curves, then the tangent functor will play a dominant role since $\gamma'(t)$ is a tangent vector, and if the theory is rather based on functions, then T^* will play a dominant role since $df(x)$ is a cotangent vector. Under the influence of contemporary algebraic geometry, most modern theories adopt the second point of view. By choosing the first one, we do not want to take a decision on "correctness" of one point of view or the other, but, being interested in the "covariant theory", we just simplify our life and avoid technical problems such as double dualization and topologies on dual spaces which inevitably arise if one tries to develop a theory of the tangent functor in an approach based on scalar functions. This allows to drop almost all restrictions on the model space in infinite dimension and opens the theory for the use of base fields of positive characteristic and even of base rings (giving the whole strength to the point of view of scalar extensions).

However, although we do not define, e.g., tangent vectors as differential operators acting on functions, they may be *represented* by such operators. Similarly, connections may be represented by covariant derivatives of sections, and so on. In other words, one can develop a *representation theory*, representing geometric objects from the "covariant setting", in a contravariant way, by operators on functions and sections – some sketchy remarks on the outline of such a theory are given in Chapter 20, the theory itself remains to be worked out. In Chapter 21, we define a particularly important differential operator, namely the *exterior derivative* of forms.

V. Lie Theory

0.9. *The Lie algebra of a Lie group.* By definition, *Lie groups* are manifolds with a smooth group structure (Chapter 5). Then, as in the classical real case, the Lie algebra of a Lie group can be defined in various ways:

(a) one may define the Lie bracket via the bracket of (left or right) invariant vector fields (as is done in most textbooks), or
(b) one may express it by a "Taylor expansion" of the group commutator $[g,h] = ghg^{-1}h^{-1}$ (as is done in [Bou72] and [Se65]; the definition by deriving the adjoint representation $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$ is just another version of this).

For convenience of the reader, we start with the first definition (Chapter 5), and show (Theorem 23.2) that it agrees with the following version of the second definition: as in the classical theory, the tangent bundle TG of a Lie group G carries again a canonical Lie group structure, and by induction, all higher order tangent bundles T^kG are then Lie groups. The projection $T^kG \to G$ is a Lie group homomorphism, giving rise to an exact sequence of Lie groups

$$1 \;\to\; (T^kG)_e \;\to\; T^kG \;\to\; G \;\to 1\,. \tag{0.5}$$

For $k = 1$, the group $\mathfrak{g} := T_eG$ is simply the tangent space at the origin with vector addition. For $k = 2$, the group $(T^2G)_e$ is no longer abelian, but it is two-step nilpotent, looking like a Heisenberg group: there are three abelian subgroups

$\varepsilon_1\mathfrak{g}, \varepsilon_2\mathfrak{g}, \varepsilon_1\varepsilon_2\mathfrak{g}$, all canonically isomorphic to $(\mathfrak{g}, +)$, such that the commutator of elements from the first two of these subgroups can be expressed by the Lie bracket in $\mathfrak{g}$ via the relation

$$\varepsilon_1 X \cdot \varepsilon_2 Y \cdot (\varepsilon_1 X)^{-1} \cdot (\varepsilon_2 Y)^{-1} = \varepsilon_1\varepsilon_2[X, Y]. \tag{0.6}$$

0.10. *The Lie bracket of vector fields.* Definition (a) of the Lie bracket in $\mathfrak{g}$ requires a foregoing definition of the Lie bracket of vector fields. In our general framework, this is more complicated than in the classical context because we do not have a faithful representation of the space $\mathfrak{X}(M)$ of vector fields on M by differential operators acting on functions (cf. Section 0.8). It turns out (Chapter 14) that a conceptual definition of the Lie bracket of vector fields is completely analogous to Equation (0.6), where $\mathfrak{g}$ is replaced by $\mathfrak{X}(M)$ and $(T^2G)_e$ by the space $\mathfrak{X}^2(M)$ of smooth sections of the projection $TTM \to M$. The space $\mathfrak{X}^2(M)$ carries a canonical group structure, again looking like a Heisenberg group (Theorem 14.3). Strictly speaking, for our approach to Lie groups, Definition (a) of the Lie bracket is not necessary. The main use of left- or right-invariant vector fields in the classical theory is to be integrated to a flow which gives the exponential map. In our approach, this is not possible, and we have to use rather different strategies.

0.11. *Left and right trivializations and analog of the Campbell-Hausdorff formula.* Following our general philosophy of multilinear connections, we try to linearize $(T^kG)_e$ by introducing a suitable "canonical chart" and to describe its group structure by a multilinear formula. This is possible in two canonical ways: via the zero section, the exact sequences (0.5) split, and hence we may write T^kG as a semidirect product by letting act G from the right or from the left. By induction, one sees in this way that $(T^kG)_e$ can be written via left- or right trivialization as an iterated semidirect product involving $2^k - 1$ copies of $\mathfrak{g}$. Relation (0.6) generalizes to a general "commutation relation" for this group, so that we may give an explicit formula for the group structure on the space $(T^k\mathfrak{g})_0$ which is just a certain direct sum of $2^k - 1$ copies of $\mathfrak{g}$. The result (Theorem 24.7) may be considered as a rather primitive version of the Campbell-Hausdorff formula. It has the advantage that its combinatorial structure is fairly transparent and that it works in arbitrary characteristic, and it has the drawback that inversion is more complicated than just taking the negative (as in the Campbell-Hausdorff group chunk). It should be interesting to study the combinatorial aspects of this formula in more detail, especially for the theory of Lie groups in positive characteristic.

0.12. *The exponential map.* As already mentioned, one cannot construct an exponential mapping $\exp : \mathfrak{g} \to G$ in our general context. However, the groups $(T^kG)_e$ have a nilpotent Lie algebra $(T^k\mathfrak{g})_0$, and thus one would expect that a polynomial exponential map $\exp_k : (T^k\mathfrak{g})_0 \to (T^kG)_e$ (with polynomial inverse $\log_k$) should exist. This is indeed true, provided that $\mathbb{K}$ is a field of characteristic zero (Theorem 25.2). Our proof of this fact is purely algebraic (Appendix PG, Theorem PG.6) and uses only the fact that the above mentioned analog of the Campbell-Hausdorff formula defines on $(T^kG)_e$ the structure of a *polynomial group* (see Section 0.18 below). By restriction to the jet bundle J^kG, we get an exponential map $\exp_k$ for the nilpotent groups $(J^kG)_e$, which form of projective system so that, as projective limit, we get an exponential map $\exp_\infty$ of $(J^\infty G)_e$ (Chapter 32). If there is a notion of convergence such that $\exp_\infty$ and $\log_\infty$ define convergent series,