Edited by
G. Denardo and H. D. Doebner

Conference on Differential Geometric Methods in Theoretical Physics

International Centre for Theoretical Physics, Trieste
30 June - 3 July 1981

World Scientific

World Scientific Publishing Co Pte Ltd
P O Box 128
Farrer Road
Singapore 9128

ISBN 9971-950-58-8

Printed in Singapore by Richard Clay (S. E. Asia) Pte. Ltd.

FOREWORD

The Conference on Differential Geometric Methods in Theoretical Physics was held at the International Centre for Theoretical Physics in Trieste from 30 June to 3 July 1981. The directors were G. Denardo (University of Trieste) and H.D. Doebner (University of Clausthal, F.R.G.).

Besides the International Centre for Theoretical Physics, the International School for Advanced Studies in Trieste and the Office for Foreign Studies and Activities at the Technical University of Clausthal (Federal Republic of Germany) have given their contribution to the success of the conference. We are presenting here the lectures and seminars which have been given.

The Conference has been organized in the spirit of the tradition of the earlier conferences on the same theme held in Aix-en-Provence, Bonn, Warsaw, Clausthal and Salamanca. While focusing primarily upon recent advances in geometrical and topological aspects of field theories, non-Abelian gauge theories, supersymmetries, general relativity and quantization methods, the main aim was that of bringing together physicists and mathematicians representing a wide spectrum of interests.

The directors of the Conference wish to thank particularly Professor Abdus Salam, Director of the International Centre for Theoretical Physics, Trieste, and Professor Paolo Budinich, Director of the International School for Advanced Studies in Trieste for their essential scientific, moral and material support to the Conference.

G. Denardo   H.D. Doebner

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I. GENERAL REVIEW LECTURES

ADVANCES IN SUPERGRAVITY

Yuval Ne'eman*,**
Tel Aviv University*** and University of Texas, Austin†,††,†††

1. The Physics of Supergravity

In recent years, General Relativity has re-entered Physics, with observations and theoretical developments influencing each other. Steady-State Theory was rejected following the discovery of the $3^{\circ}K$ Background Radiation; an active search for Black Holes is taking place in connection with the new understanding of Black Hole Physics; "Grand-Unification Gauge Theories" (GUTs) have been applied to Cosmogony, providing new (speculative) explanations for the matter-antimatter asymmetry and for the 10^8 ratio of photons to baryons in the Universe.

By its very nature, Supergravity should (if successful) provide in the long run the keys to the early stages of Cosmogony. Supergravity is an embedding of Einstein's theory in a broader framework, with two main aims: finiteness of Quantum Gravity and (Super) unification with other interactions. This is Highest Energy Physics, i.e. at energies of the order of the Planck mass (10^{19} GeV), 4-5 orders of magnitude beyond GUT physics. It should dominate the creation of matter by the gravitational field, and the emergence of macroscopic asymmetries such as those studied by GUTs. Indeed, for all we know, it may even altogether

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* Mortimer and Raymond Sackler Institute of Advanced Studies, TAU

** Wolfson Chair Extraordinary, Department of Physics and Astronomy, TAU

*** Supported in part by the U.S. - Israel Binational Science Foundation

† Center for Particle Theory, UT

†† also Center for Theoretical Physics, UT

††† Supported in part by the U.S. DOE under Grant EY-76-S-05-3992

supersede the GUTs or absorb them, rather than just supplementing them above 10^{15} GeV.

2. Progress and Key Results

The key achievements to date have consisted of the following items:

(a) 2-loop finiteness of Quantum Gravity, including the presence of matter - provided that this matter, together with the gravitational field, form an appropriate "extended" supermultiplet.

(b) 3-loop super-renormalizability of a "simple theory" (N=4 Super-Yang Mills), resembling N=8 Extended Supergravity.

(c) vanishing of geometrical counter-terms for n loops, $n < N$.

(d) positive-definiteness of the Gravitational Hamiltonian.

(e) the removal of the Johnson-Sudarshan and other inconsistencies, which used to plague spin 3/2 theories.

All of these developments appear to augur well for a future finite Quantum Gravity. Indeed, they represent great advances since 1974, i.e. since the pioneer results of Veltman and 't Hooft in covariant quantization. In addition, Einstein's Unification Program has made tremendous progress. His dream of Gravity-Electromagnetism Unification has indeed been realized "better" than in any of his own attempts or those of his contemporaries. It is in this context, for instance, that item (e) has been obtained: e.g. a removal of the acausal (faster than light) propagation (the "Velo-Zwanziger paradox") "traditionally" connected with charged spin 3/2 fields. This is important, but does not represent the exalted aims which motivated Einstein in his Search. Our target has changed, and we do not consider that we have reached the goal by writing down a consistent and non-trivial Unification of Gravity with Electromagnetism ("N=2 Extended Supergravity"). Instead we have pursued the program all the way to N=8, the widest Superunification allowed by field-theory as we know it; it is also a particularly promising framework for a finite theory.

Almost every step from N=1 to N=8 Supergravity has brought insights, and I shall devote much of this review to the N=8 theory, including mention of attempts to provide it with a phenomenological interpretation.

3. Supersymmetry

The Super-Poincaré algebra $\hat{p}$ was introduced[1)] by Golfand and Likhtman in 1971[†]. Wess and Zumino worked out[2)] in 1973 the transformations defining the Super conformal group $\hat{C} \supset \hat{P}$. Volkov and Soroka[3)] and Salam and Strathdee[4)] introduced superspace, the factor manifold $\hat{S} = \hat{P}/L$, where L is the Lorentz group $SL(2\,\mathbb{C})$. Corwin, Ne'eman and Sternberg[5)] clarified the structure of a Lie super-algebra (or graded Lie algebra, GLA) and defined the Extended Superconformal group $\hat{C}_N$ and its subgroup $\hat{P}_N$, known as Extended Supersymmetry. Haag, Lopuszanski and Sohnius[6)] showed that $\hat{P}_N$ and $\hat{C}_N = SU(2, 2/N)$ but[7,8)] $MSU(2, 2/4) = SU(2, 2/4)/I = \hat{C}_4$, I denoting the identity provide the only possibility to unify internal symmetries with those of space-time.

$\hat{c}_N$ is $\mathbb{Z}$ graded, with the grading corresponding to a doubling of the dimension (i.e. of the bracket-eigenvalue of Δ, the dilation generator).

$$\begin{array}{ccccccccc}
\ldots & c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & c_3 & \ldots \\
\ldots & 0 & \Xi_\mu & \Psi^a_\alpha & \Sigma_{\mu\nu} & T^a_\alpha & \Pi_\mu & 0 & \ldots \\
 & & & & \Delta & & & & \\
 & & & & \Omega_{ab} & & & & \\
 & & & & \Theta_{ab} & & & &
\end{array} \qquad (3.1)$$

$\mu,\nu = 0,1,\ldots 3$; $a,b = 1,2,\ldots N$ $\quad$ ¦__displays $\hat{p} \subset \hat{c}_N$

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† we use capital letters for groups, lower case type for algebras, gothic for manifolds, and a caret for "super", or "graded" in the sense of Ref. 5.

$\ell := \{\Sigma_{\mu\nu}\} = s\ell(2,\mathbb{C})$ is the Lorentz algebra, π_μ are the translations, Ξ_μ the special conformal transformations; $r := \{\Omega_{ab}\} = so(N,\mathbb{R})$ generates an orthogonal group; $\{\Theta_{ab}\} = u(N)/r$ is an odd-parity set of generators forming (together with the Ω_{ab}) the algebra of $u(N)$ (except for the case $N=4$, where it generates $SU(4)$). For $N=1$, there is no Ω_{ab}, but Θ_{00} exists as $\Theta = \gamma_5$, the chiral transformations. The odd-grade generators obey the brackets,

$$
\begin{aligned}
\{T^a_\alpha\ ,\ T^b_\beta\} &= -\frac{1}{2}\,\delta^{ab}\,(\gamma^\mu C)_{\alpha\beta}\,\pi_\mu \\
\{\Psi^a_\alpha\ ,\ \Psi^b_\rho\} &= \frac{1}{2}\,\delta^{ab}\,(\gamma^\mu C)\,\Xi_\mu \\
\{T^a_\alpha\ ,\ \Psi^b_\beta\} &= -\frac{1}{2}\,\delta^{ab}\,C_{\alpha\beta}\,\Delta + \delta^{ab}\,(\sigma^{\mu\nu}C)_{\alpha\beta}\,\Sigma_{\mu\nu} + C_{\alpha\beta}\,\Omega^{ab} \\
&\quad + i(\gamma_5 C)_{\alpha\beta}\,\Theta^{ab} \qquad (3.2)
\end{aligned}
$$

We refer the reader to references[5,6,9)] for the remaining brackets, listing here only those that will be needed for the sequel,

$$
\begin{aligned}
[T^a_\alpha\ ,\ \Sigma_{\mu\nu}] &= (\sigma_{\mu\nu})^\beta_\alpha\,T^a_\beta \\
[T^a_\alpha\ ,\ \Omega_{bc}] &= \delta^a_b\,T^b_\alpha - \delta^a_c\,T^c_\alpha \\
[T^a_\alpha\ ,\ \Pi_\mu] &= 0 \\
[\Xi_\mu\ ,\ \Pi_\nu] &= -2(\eta_{\mu\nu}\Delta + \Sigma_{\mu\nu}) \qquad (3.3)
\end{aligned}
$$

Note that T^a_α and Ψ^a_α are "real" Majorana spinors, existing only in space-time dimensionalities which allow the γ^μ to take on pure imaginary values. This happens in 2,3,4 modulo 8 dimensions[10)]. In these dimensionalities $d=4+N$, an $so(1, N+3)$ spinor has $2^{[(N+3)/2]}$ components, where [ ] denotes the integer part. Such spinors in $4+N$ dimensions reduce to $2^{[(N-1)/2]}$ real Majorana space-time 4-spinors.

Returning to (3.1), Superspace is the factor manifold

$$\hat{S}_N = \hat{P}_N/L \times R$$

where L is the Lorentz group manifold and R that of the $SO(N,\mathbb{R})$ internal orthogonal symmetry.

4. Gauging on a Soft Group Manifold

In Supergravity[11,12] superspace $\hat{S}_N(\chi^\mu, \Theta^{\alpha a})$ is curved and all brackets (3.3) and (3.4) etc. may acquire additional terms on the right hand side (:= and =: represent a definition)

$$[x_{curved}, y_{curved}] := [x,y] + \sum_A R^A_{xy}\, z_A \tag{4.1}$$

where R^A_{xy} is thereby a "curvature" by definition. As I showed with T. Regge[13)], and developed with J. Thierry-Mieg[14)] (see also Ref. 15), we are dealing here with a "Soft-Group-Manifold" (SGM) undergoing a fibration as a result of the equations of motion. The SGM is a triplet $G:=(G,\mathcal{G},\rho)$ where G is the non-internal group (such as P, $\hat{P}$, $\hat{P}_N$ etc...), $\mathcal{G}$ is a differentiable manifold of the same dimensionality "m", and ρ^A is a set of frames over $\mathcal{G}$, behaving under G as its adjoint representation. Explicitly this is a one-form

$$\rho^A = \rho^A_M\, d\,\zeta^M \quad , \tag{4.2}$$

where ζ^M is the coordinate over $\mathcal{G}$. Picking an action

$$S = \int_{R^d} R^A \wedge T_A \quad , \tag{4.3}$$

R^d is Riemannian d-dim, spacetime, $R^A := R^A_{MN}\, d\zeta^M \wedge d\zeta^N$ and T_A are respectively 2-forms and (m-2) forms. The T_A are chosen so as to break G and preserve local F-symmetry, $F \subset G$. For instance, $F := ((\Sigma_{\mu\nu}, \Omega_{ab})) = s\ell(2,\mathbb{C}) \otimes so(N)$ in $\hat{P}_N$. The equations of motion enforce in $\mathcal{G}$ a fibration ($\overset{\Omega}{=}$ denotes equality "on mass shell")

$$\overset{F}{\mathcal{G}} :\overset{\Omega}{=} \mathcal{G}\ , \quad \overset{F}{\mathcal{G}}\ (\overset{F}{\mathcal{G}},\ F,\ \mathcal{G}/F,\ \pi,\ \cdot) \tag{4.4}$$

and R^A or T_A become constrained as forms to be "horizontal" $H \overset{\Omega}{=} \mathcal{G}/F$,

$$F \overset{\Omega}{:=} \mathcal{F}$$

which is in fact superspace $\hat{S}_N$. The fibration implies ({F} represent the range of indices of F)

$$B \text{ or } C \in \{F\} \;, \rightarrow R^A{}_{BC} \overset{\Omega}{=} 0 \quad . \tag{4.5}$$

Besides the above result, we also find that the generalized torsion vanishes

$$R^A \overset{\Omega}{=} 0, \quad \forall \{A\} \in G/F \quad . \tag{4.6}$$

A sufficient condition[13,14] for G to fibrate, for the relevant groups[14] G is that they be weakly-reducible,

$$\left.\begin{array}{l} g = f \oplus h \\ [f,f] \subset f \\ [f,h] \subset h \end{array}\right\} \tag{4.7}$$

and in addition have a symmetric decomposition

$$[h,h] \subset f \quad . \tag{4.8}$$

The latter condition is not fulfilled by the (3.3) commutator $\{T\ T\} \subset \Pi$, since both T and Π are in h, for $F = ((\Sigma))$, $g = \hat{p}$. Supergravity can be regarded either as an incomplete fibration, or, for $N = 1$, the action can be represented by the sum of two fibrated G^+ and G^-, where f is chosen as[13,14]

$$f^{+}_{-} = \ell \oplus ((T^{+}_{-})) \tag{4.9}$$

and T^{+}_{-} are the Left and Right Chiral components. As a result,

$$\left.\begin{array}{ll} R^i \overset{\Omega}{=} 0 & i,j = 0,1..3 \\ R^A_{[ij]B}\, \rho^{[ij]} \wedge \rho^B \overset{\Omega}{=} 0 & a,b = 1...N \\ R^A_{[ab]B}\, \rho^{[ab]} \wedge \rho^B \overset{\Omega}{=} 0 & [ij] \in \{s\ell(2\mathbb{C})\} \\ R^i_{\alpha a,\beta b} \overset{\Omega}{=} 0 & [ab] \in \{so(N)\} \end{array}\right\} \tag{4.9}$$

and α and β are Majorana spin indices.

The "curved" generators of equation (4.1) are given by an orthonormal basis $\tilde{a}$ of "vector fields" (mathematical appelation) in the Soft Group Manifold G

$$\rho^A \lrcorner\, \tilde{a}_B = \delta^A_B \quad . \tag{4.10}$$

The $\tilde{a}_i$ and $\tilde{a}_\alpha$ are in fact the corresponding "covariant derivatives" and when locally gauged, correspond to the "Lie derivative" by an anholonomic vector field $\tilde{\varepsilon}^B$

$$\delta\rho^A = L_{\tilde{\varepsilon}}\, \rho^A = D\varepsilon^A + \varepsilon^B \wedge \rho^C\, R^A{}_{BC} \tag{4.11}$$

"Local supersymmetry", the transformation generating Supergravity, indeed corresponds to $\bar{\varepsilon}_\alpha{}^a$ in (4.11). It is an "anholonomized" general coordinate transformation, and we shall derive the variations for the various gauge fields. However, we can already note that the last equation in (4.9) guarantees

$$\{\tilde{T}^a_\alpha, \tilde{T}^b_\beta\} \overset{o}{=} -\frac{1}{2}\delta^{ab}\,(\gamma^\mu\, \tilde{\pi}_\mu)$$

which, when applied to a state at rest, yields on mass shell

$$\{\tilde{T}^a_\alpha, T^b_\beta\}\ |\vec{p} = 0 > \overset{o}{=} \frac{1}{2}\,\delta^{ab}\, \tilde{\Pi}_0 |\ \vec{p} = 0 > \tag{4.12}$$

where $\tilde{\Pi}_0$ is the Hamiltonian (or the supercovariant derivative). This can be shown to guarantee[16)] positive-definiteness of the gravitational Hamiltonian, even after a gradual extinction of the "super" contributions. However, (4.12) is written in superspace $\hat{S}(x^\mu, \Theta^{\alpha a})$ and when we restrict to spacetime, the algebra does not close[17)].

Note that Ω_{ab} in (3.2) disconnects from the rest of $\tilde{P}_N$. It acts on the T^a_α, which behave under it as an $SO(N)$ vector, but it is not reproduced by a commutator or anticommutator of $\tilde{P}_N/SO(N)$. As a result, it is not gauged locally with the factor group $\tilde{P}_N/SO(N)$.

An alternative approach involves working on $\hat{S}_N$ off-mass-shell as well as on-mass-shell. The action cannot correspond to (4.3) and has generally been taken to be identical with the measure over superspace[18], with appropriate constraints[19-20]. This "direct" superspace (DS) method (or an equivalent set) has been applied to N=1 and higher N. For N=1, the $\bar{\varepsilon}^{\alpha}$ variations have been closed off-shell by identifying the appropriate curvature components (4.1) with new superfields (fields in χ, Θ).

For quantum Gravity to be finite (it cannot be just renormalizable, due to the dimensional Newtonian coupling κ) within a renormalizable quantum Supergravity written over $\hat{S}_N$ (and not G), we have to prove that no counter-terms can appear at any order.

5. Extended Supergravity Representations

Note the δ^{ab} on the right-hand side of (4.12); the coupling of internal and external degrees of freedom is non-trivial. This is why Supergravity is an advance over the Kaluza-Klein solution. Over there, the addition of a (compactified) dimension produces an elegant derivation of the Einstein-Maxwell theory, but with no new mathematical constraints. It is just a juxtaposition, (not a coupling) of the two interactions. We had generalized it[21,22] to $su(3) \subset su(4) = \overline{so}(6)$ by embedding $R^4 \subset M^{10}$, in an early attempt to renew the search for Unification, hypothetizing that "flavor" symmetries emerge through the short-range character of the Strong Interactions. In Extended Supergravity, however, equation (4.12) couples the internal degrees of freedom to the external.

Particle states can be defined in a local frame. We then have the representations of $\hat{P}_N$, given by the little group. For (3.3) acting on massive states[23], we have (4.12) which defines a 4N-dimensional Clifford algebra. For N=1 the 4 states are given by $(s, s+\frac{1}{2}, s-\frac{1}{2}, s')$, where s is some basic helicity $\Sigma_{12} = f_3 = s$. For massless states (or for $\hat{C}_N$), we use[5] $p^{\pm} = p_0 \pm p_3$. The system reduces into $p^+ \neq 0$, $p^- = 0$ and vice versa. At $p^{\pm} \to \infty$, this is identical with R/L chiralities.

The massless representations of Supergravity,[24] for N=1 are 2-dimensional (helicities f_3 = s, s-½) but CPT invariance requires a doubling, adjoining (-s, -s + ½). For N, the number of states at each helicity $j_3 = j_3^{max} - k/2$ (using lowering operators $T^-_{\alpha a}$) in an irreducible representation is given by the binomial coefficient $\binom{N}{k}$, as befits Clifford or Grassmann algebras in which the anticommutator is a number (or zero), and only the antisymmetrized combination contributes to the enveloping algebra. Symbolically, the representation comprises in terms of Young tableaux,

$$|j_3^{max}> \oplus \square^{a}_{\Delta j_3 = -\frac{1}{2}} |j_3^{max}> \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} |j_3^{max}> \oplus \dots \oplus \begin{array}{|c|l}\hline & 1 \\ \hline & 2 \\ \hline & \vdots \\ \hline & N \\ \hline \end{array} |j_3^{max}> \tag{5.1}$$

For N=1 we get helicities (2; 3/2) ⊕ (-3/2; -2), the spin 2 graviton and spin 3/2 "gravitino". For N=2 we have (2; 2 x 3/2; 1) ⊕ (-1; 2 x (-3/2); -2). This is the realization of Einstein's aim: a graviton, the electromagnetic field with spin 1, and a charged spin 3/2 gravitino. The system continues to grow all the way to N=7,

(2; 7x3/2; 21x1; 35x½; 35x0; 21x(-½); 7x(-1); -3/2; -)

⊕ (-; 3/2; 7x1; 21x½; 35x0; 35x(-½); 21x(-1); 7x(-3/2); -2)

(5.2)

whereas for N=8 we have one irreducible representation, though with the same particle-state composition as N=7

(2; 8x3/2; 28x1; 56x½; 70x0; 56x(-½); 28x(-1); 8x(-3/2); 2) .

(5.3)

Note that though the spin 1 states do fit the adjoint representation of SO(N), they do not correspond to the gauge fields of local SO(N), which does not become gauged in $\hat{P}_N$. In fact, all fields in these representations correspond to a "2nd order" formalism and are not the direct (1st order) connections. They are related to them by equations of motion, constraints or Bianchi identities, depending on the formalism. This is just as in Gravity, where the vanishing of the torsion $R^i = 0$ is used to solve $\rho_\mu^{[ij]}$ (the Lorentz connection) in terms of $\rho_\mu^{\ i}$ (the tetrad) and reproduce the Christoffel formula. The $j = 3/2$ field in the representation is related to $\rho_\mu^{\ \alpha a}$ in a complicated way, since $j = 2$ stands for $g_{\mu\nu}$ rather than $\rho_\mu^{\ i}$ or $\rho_\mu^{[ij]}$

The irreducibility of N=8 is typical of systems in which $f_3^{max} = N/4$. One such theory is the "special" N=4 Yang-Mills supersymmetry[25,26] with

$$(1;\ 4\times\tfrac{1}{2};\ 6\times 0;\ 4\times(-\tfrac{1}{2});\ -1) \tag{5.4}$$

This theory has recently[27-28] been shown to have vanishing coupling constant renormalization[29] $\beta = 0$ to 3 loops,

$$\lambda \frac{dg(\lambda)}{\lambda} = \beta(g(\lambda)) \tag{5.5}$$

These irreducible particle representations of N=extended supersymmetry also have in common the fact that their particle content is the same as that of the reducible N-1 case.

The $\hat{P}_N$ representations are also representations[30] of su(1/N), as can be checked by comparing with Refs. (31)-(33). Since the particles are massless, they obey the superconformal algebra su(2, 2/N). The latter's representations are induced by the "little group".

For the irreducible $N = 4f_3^{max}$ cases, we can extend SU(1/N) by the discrete CPT, since the representations are also eigen-representations of CPT. We may thus define a super-unitary parity[30] η_s, taking for instance the CPT parity of the $f_3 = 0$ symmetric combination of the "middle" states.

The $\beta(N=4) = 0\big|_{n \leq 3}$ result may augur well for a finite theory. It is possible that the theory is indeed finite to all orders, and that the same be true of N=8 supergravity!