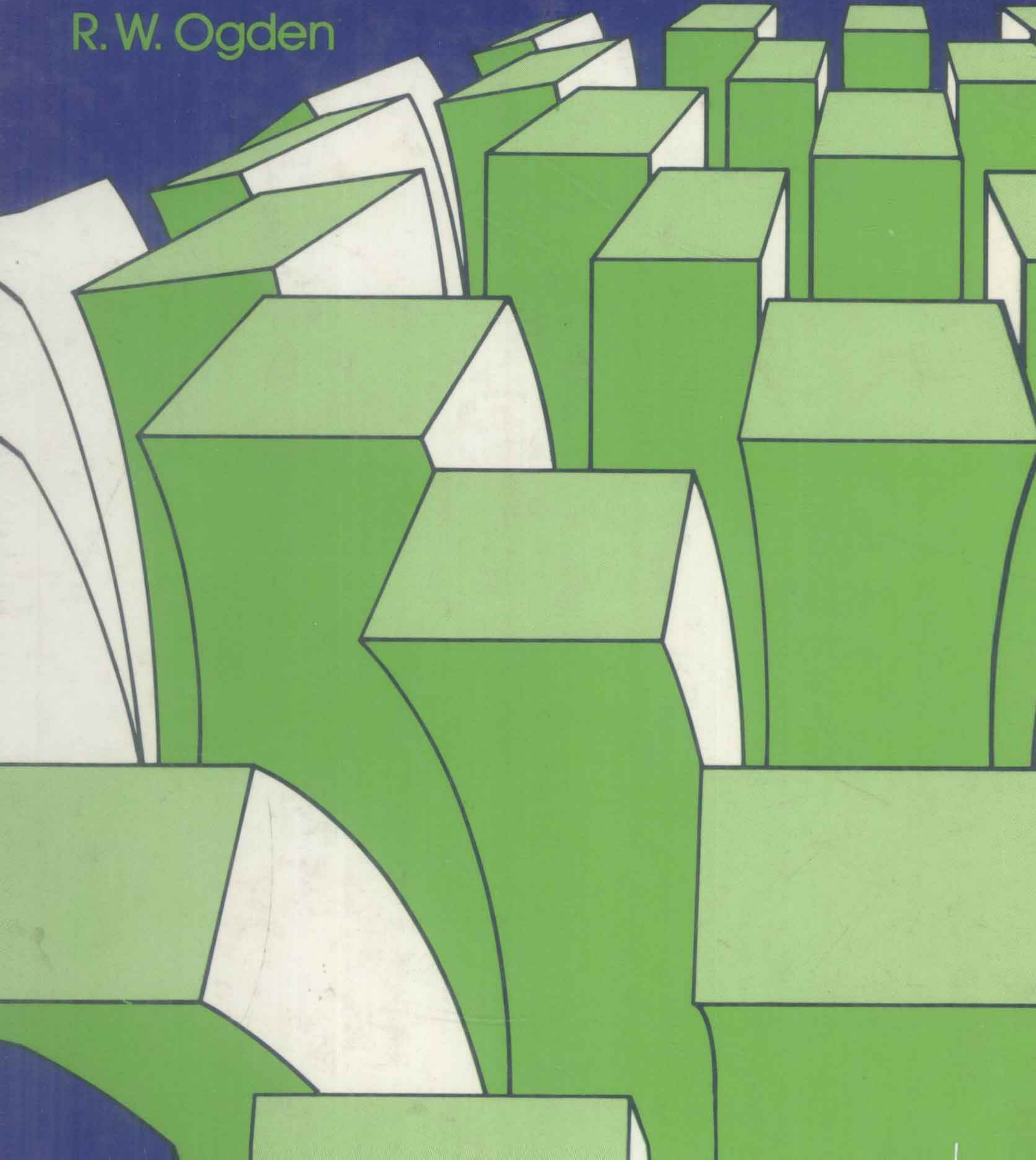
Ellis Horwood Series MATHEMATICS AND ITS APPLICATIONS

# NON-LINEAR ELASTIC DEFORMATIONS

R. W. Ogden



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## NON-LINEAR ELASTIC DEFORMATIONS

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This meticulous and precise account of the theory of finite elasticity fills a significant gap in the literature. It covers the application of this theory to the solution of boundary-value problems, and to the analysis of the mechanical properties of solid materials capable of large elastic deformations. It is especially coherent and well written, and will surely come to be regarded as a classic treatment of the topic.

In addition to the inclusion of all the basic material, the author contributes many unpublished results and provides new approaches to existing problems. Thus the book can be regarded both as a research work and as a text suitable for advanced post-graduate study.

The author's authority in this field is reflected in his distinctive approach. He has designed problems to develop the text material, some of them containing statements of new results. The setting is purely isothermal, with attention confined largely to the quasi-static theory, although some relevant discussion of time-dependent problems is included.

The publishers know of no such comprehensive book to rival this work; few cover the field so carefully and thoroughly. It concentrates on "exact" theories in the sense that no discussion of, for example, rod or shell theories, is included. Within this framework, a broad spectrum of topics is covered, and a balanced overview achieved.

**Readership:** Research workers and research students in applied mathematics, mechanical engineering, and continuum mechanics. Also physicists, materials scientists, and other scientists concerned with the elastic properties of materials.

**R. W. Ogden** is Professor of Mathematics at Brunel University. He was previously Reader in Mathematics at the University of Bath (1976–1980); a Lecturer in Mathematics at the same university from 1972–1976; and was SRC Research Fellow at the University of East Anglia (1970–1972). He gained a B.A. in Mathematics in 1966 and a Ph.D. in Solid Mechanics in 1970, both from the University of Cambridge.

# NON-LINEAR ELASTIC DEFORMATIONS

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To
My parents

# Preface

This book is concerned with the mathematical theory of non-linear elasticity, the application of this theory to the solution of boundary-value problems (including discussion of bifurcation and stability) and the analysis of the mechanical properties of solid materials capable of large elastic deformations. The setting is purely isothermal and no reference is made to thermodynamics. For the most part attention is restricted to the quasi-static theory, but some brief relevant discussion of time-dependent problems is included.

Apart from much basic material the book includes many previously unpublished results and also provides new approaches to some problems whose solutions are known. In part the book can be regarded as a research monograph but, at the same time, parts of it should also be suitable as a postgraduate text. Problems designed to develop further the text material are given throughout and some of these contain statements of new results.

Because so much of the theory depends on the use of tensors, Chapter 1 concentrates on the development of much of the tensor algebra and analysis which is used in subsequent chapters. However, there are parts of the book (in particular, Sections 4.4 and 7.2) which do not rely on a knowledge of tensors and can be read accordingly. Chapter 2 provides a detailed development of the basic kinematics of deformation and motion. Chapter 3 deals with the balance laws for a general continuum and the concept of stress. Prominence is given to the nominal stress tensor and the notion of conjugate stress and strain tensors is examined in detail.

In Chapter 4 the properties of the constitutive laws of both Cauchy- and Green-elastic materials are studied and, in particular, the implications of objectivity and material symmetry are assessed. Considerable attention is devoted to isotropic constitutive laws for both (internally) constrained and unconstrained materials. The basic boundary-value problems of non-linear elasticity are formulated in Chapter 5 and the governing equations are solved for a selection of problems in respect of unconstrained and incompressible isotropic materials. A section dealing with variational aspects of boundary-value problems is included along with a short discussion of conservation laws.

Chapter 6, the longest chapter, is concerned with incremental deformations superposed on an underlying finite deformation. The resulting (linearized) boundary-value problem is formulated and its structure discussed in relation to the analysis of uniqueness, stability and bifurcation. The role of the strong ellipticity inequality is examined. Constitutive inequalities are discussed and the implications of their failure in relation to bifurcation (or branching) is assessed from the local (i.e. incremental) viewpoint. Global aspects of non-uniqueness are also considered. The incremental theory is then applied to some representative problems whose bifurcation behaviour is studied in detail.

In the final chapter, Chapter 7, the theory of elasticity is applied to certain deformations and geometries associated with simple experimental tests, in particular the pure homogeneous biaxial deformation of a rectangular sheet. The relevant theory is provided in a concise form as a background for comparison with experimental results, isotropic materials being considered for simplicity of illustration. This is then used to assess the elastic response of certain rubberlike materials. The incremental theory governing the change in deformation due to a small change in material properties is developed and applied to the case of a slightly compressible material and this in turn is illustrated by means of rubberlike materials.

The book concentrates on 'exact' theories in the sense that no discussion of 'special' theories, such as shell, rod or membrane theories, or of numerical methods is included. (Excellent separate accounts of these topics are available elsewhere.) Within this framework a broad spectrum of topics has been covered and a balanced overview attempted (although this is, not surprisingly, influenced by the areas of the subject on which the writer has been actively engaged). Attention is confined to twice-continuously differentiable deformations on the whole, with discontinuities being touched on only briefly, in Chapter 6, in relation to failure of ellipticity.

References to standard works for background reading are given throughout the text but historical attributions and detailed lists of references to papers are not provided. Only where further development of the textual material might be required are references to the more recent papers cited, but the list of references is not intended to be exhaustive. References are indicated by the author's name followed by the year of publication in the text and gathered together at the end of each chapter.

# Acknowledgements

The content and style of the book owes much to the influence of many friends and colleagues over several years, but I should like to record my particular gratitude to Professor Rodney Hill, FRS, University of Cambridge, who introduced me to the subject of non-linear elasticity when I was a research student (1967–70), and Professor Peter Chadwick, FRS, University of East Anglia, with whom I spent two years as a research fellow (1970–72).

Thanks are due to Drs. Gareth Parry and Keith Walton, University of Bath, who undertook the task of reading and criticizing the whole of the manuscript, to Michael Warby, Brunel University, on whose calculations Figs. 6.7 and 6.9 were based, and Molly Demmar for her excellent typing. Finally, I should like to thank my publisher Ellis Horwood for his patience in waiting for the delivery of a long-overdue manuscript.

R.W. Ogden
*Brunel University*
*May* 1983

# Table of Contents

CHAPTER 1

# Tensor Theory

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The use of vector and tensor analysis is of fundamental importance in the development of the theory which describes the deformation and motion of continuous media. In non-linear elasticity theory, in particular, little progress can be made or insight gained without the use of tensor formulations. This first chapter is therefore devoted to an account of the vector and tensor algebra and analysis which underlies the requirements of subsequent chapters. Some theorems of tensor algebra, however, are not dealt with here but postponed until the later chapters in which they are needed.

It is assumed that the reader is familiar with elementary vector and matrix algebra, including determinants, with the concept of a vector space, including linear independence and the notion of a basis, and with linear mappings. Also some familiarity with the index (or suffix) notation and the summation convention is assumed. Nevertheless, certain basic ideas are summarized in the early part of this chapter, primarily to establish notations but also for convenience of reference.

## 1.1 EUCLIDEAN VECTOR SPACE

The set of real numbers is denoted by $\mathbb{R}$. A *scalar* is a member of $\mathbb{R}$.

A (real) vector space $V$ is a set of elements (called *vectors*†) such that (a) $\mathbf{u}+\mathbf{v}\in V$, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$, (b) $V$ contains the *zero vector* $\mathbf{0}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in V$ and for every $\mathbf{u}\in V$ there is an inverse element, denoted $-\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$, (c) $\alpha\mathbf{u}\in V$, $1\mathbf{u}=\mathbf{u}$, $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}$, $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$, $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$ for all $\alpha,\beta\in\mathbb{R}$, $\mathbf{u},\mathbf{v}\in V$, where 1 denotes unity.

A Euclidean vector space $\mathbb{E}$ is a real vector space such that, for any pair of vectors $\mathbf{u},\mathbf{v}\in\mathbb{E}$, there is defined a scalar, denoted $\mathbf{u}\cdot\mathbf{v}$, with the properties

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u} \tag{1.1.1}$$

$$\mathbf{u}\cdot\mathbf{u}\geq 0 \tag{1.1.2}$$

† In this chapter vectors are denoted by bold-face, lower-case letters, e.g. $\mathbf{t},\mathbf{u},\mathbf{v},\ldots$.