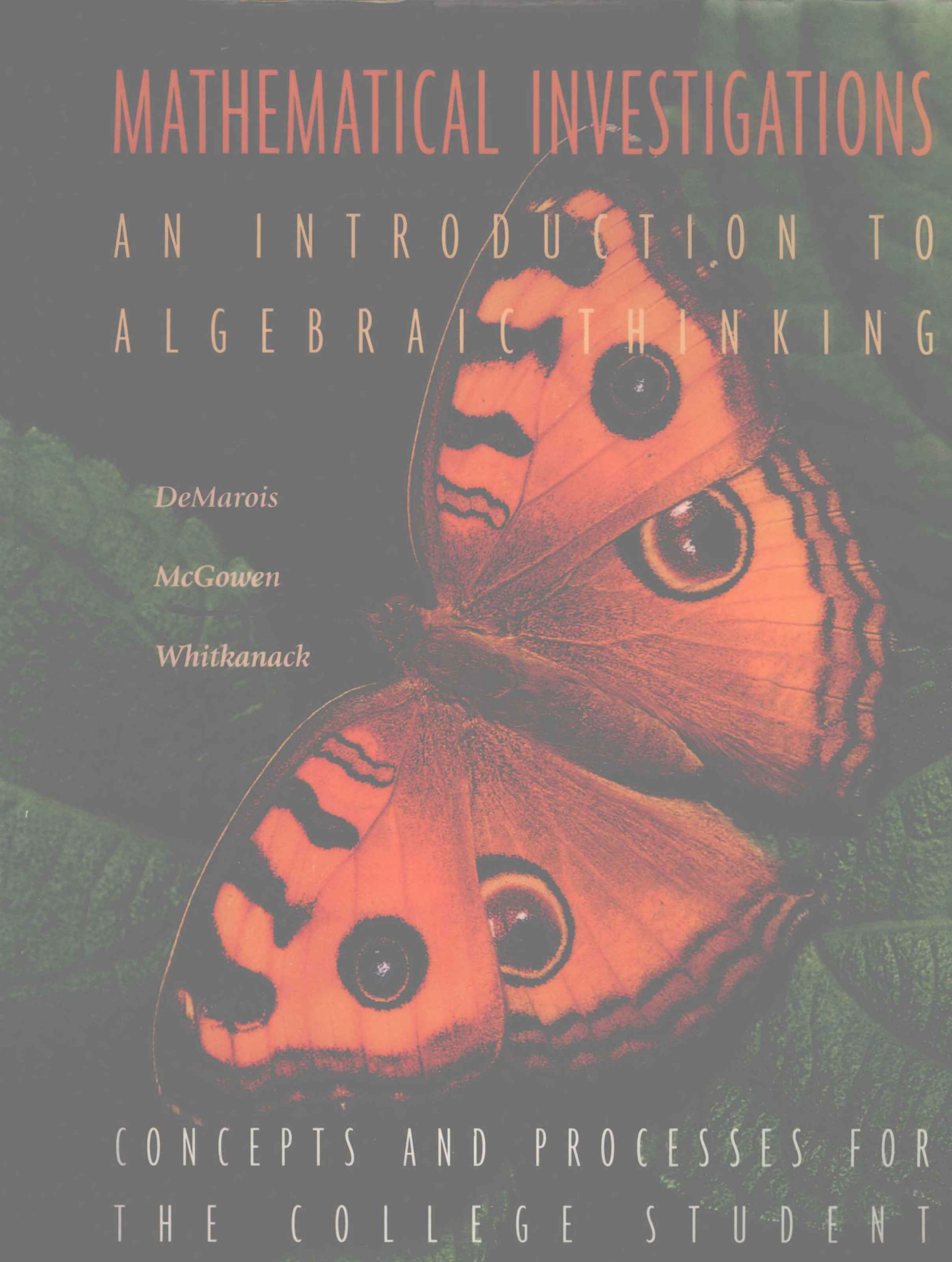
MATHEMATICAL INVESTIGATIONS

AN INTRODUCTION TO ALGEBRAIC THINKING



DeMarois

McGowen

Whitkanack

CONCEPTS AND PROCESSES FOR THE COLLEGE STUDENT

MATHEMATICAL INVESTIGATIONS

AN INTRODUCTION TO ALGEBRAIC THINKING

Phil DeMarois
William Rainey Harper College

Mercedes McGowen
William Rainey Harper College

Darlene Whitkanack
Indiana University

CONCEPTS AND PROCESSES FOR THE COLLEGE STUDENT

ADDISON-WESLEY

An imprint of Addison Wesley Longman, Inc.

Reading, Massachusetts • Menlo Park, California • New York • Harlow, England
Don Mills, Ontario • Sydney • Mexico City • Madrid • Amsterdam

Sponsoring Editor: Jason Jordan
Project Editor: Kari Heen
Production Supervisor: Kathleen Manley
Design Supervisor: Susan Carsten
Text Designer: Catherine Hawkes
Cover Designer: Judy Arisman
Prepress Services Buyer: Caroline Fell
Composition: TKM Productions
Technical Art Consultant: Joseph Vetere
Marketing Managers: Andy Fisher and Liz O'Neil
Manufacturing Supervisor: Ralph Mattivello

Cover photo © Shumway, Gail/FPG International Corporation

Partial support for this work was provided by the National Science Foundation's Course and Curriculum Development Program through grant DUE #9354471. The opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

DeMarois, Phil.
Mathematical investigations : an introduction to algebraic thinking : concepts and processes for the college student / Phil DeMarois, Mercedes McGowen, Darlene Whitkanack.
p. cm.
Includes index.
ISBN 0-321-01049-3 (pbk.)
1. Algebra. I. McGowen, Mercedes A. II. Whitkanack, Darlene. III. Title.
QA154.2.D446 1998
512.9--dc21 97-13156
CIP

1 2 3 4 5 6 7 8 9 10—CRK—0100999897

Preface

A Special Word About the First Edition

After six years of writing and testing materials, rewriting, and more testing, we look back with great respect and appreciation of the colleagues who have supported our ideas, attended our workshops, and implemented the materials in their classrooms. For us it has been a time of tremendous professional and mathematical growth as we have wrestled with translating our concerns about well-known student misconceptions into workable solutions for the classroom. The opportunity to debate our ideas and approach with reflective researchers and practitioners has resulted in the incorporation of many ideas about teaching and learning from the mathematics education community. The success of the materials is directly proportional to our ability to incorporate those ideas into these materials. The cooperation of a large number of students enrolled in the program who willingly discussed their experiences and participated in interviews that probed their understandings has resulted in materials that reflect student input as well.

In many ways our original vision has changed to reflect current theory and practice as we have attended conferences, read papers, and talked to colleagues. We have come to recognize, for example, a greater need on the part of many students for more practice to reinforce conceptual understanding than we originally believed necessary. In fundamental ways, our vision from six years ago has provided a continuing philosophical framework as the materials developed. The cover of this textbook suggests the metamorphosis we have undergone as we developed these materials. We hope that students also experience a metamorphosis in their thinking and attitudes toward mathematics as they use this text. We believe that the following descriptions reflect our philosophy and the curriculum materials we have developed.

The Course and Its Audience

This book is the first in a series that provides curriculum materials for teachers who want to make changes in the way they teach algebra. We wanted to teach an algebra course that focused on function and concepts rather than rote skills, but we found that the materials available did not meet our needs for several reasons. Sometimes they were written for more mathematically mature students than were enrolled in college developmental algebra courses.

Other materials did not fully integrate the tools we believe should become as automatic as pencil and paper to enhance the student's ability to explore mathematics. Most importantly, many of the materials seemed in conflict with mathematics education research on the teaching and learning of algebra. This book is not for the faint-hearted. Teachers who have piloted these materials report that the use of these materials has changed the way they think about mathematics and the way they teach. Knowing something about the background and philosophy of these materials may help those contemplating the use of nontraditional materials in their classroom.

The National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* provided a vision and gave direction to change the way mathematics was taught. With the inclusion of technology as an essential feature, the emphasis on problem solving, communication, reasoning, and connections empowered students in the field of mathematics and gave hope to frustrated teachers. Perhaps there was a way to make mathematics accessible to all students who were disenfranchised because of their lack of previous long-term success. Documents from the National Research Council, the Mathematical Sciences Education Board (MSEB), and the Mathematical Association of America (MAA) reinforced the need for change. We began to develop a set of materials that reflect our interpretation and agreement with ideas coming from research and working conferences. The American Mathematical Association of Two-Year Colleges (AMATYC) *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus* has served as a benchmark for our efforts.

The changes called for in these documents focus on the need for restructured curriculum materials coupled with changes in instructional philosophy and strategy. Change occurs when teachers have a plan to achieve that change and transition materials to help all students make the change confidently. Those materials must address appropriate technology use, learning styles, classroom management, assessment, and core curriculum.

An important aspect is the need for relevant content. Developmental algebra courses that meet the needs of students and industry can no longer be based solely on algorithmic techniques. Material must be presented in context. The sequence of ideas must be carefully considered. What is valued should be assessed but not necessarily graded. More emphasis should be placed on algebraic reasoning. A long-term immersion in language and structure, acquired and used over time, becomes an essential element of studying algebra. This kind of change does not occur magically. Teachers must have support and sufficient background to make these changes. Carefully planned in-service and ongoing training to refocus and support teachers in acquiring a broader view of algebra is essential.

Approach

This text promotes a pedagogical approach based on a constructivist perspective of how mathematics is learned. A student learns mathematics by working in a social context to construct mathematical ideas and by reflecting on these

constructions as a means of making sense out of problem situations. Each section of the chapters includes **investigations** of a problem situation. After gathering data, students work collaboratively on small tasks based on the investigation activities. The tasks help students reflect on mental constructions they have made as a result of working with the calculator and/or computer.

Having students talk with the members of their group as they work on the task is another way to encourage them to reflect on the problems and the solutions, whether discovered by themselves or presented by someone else in class (such as a member of another group or even the instructor).

Students are expected to answer each investigation in the text—*they write* portions of the book. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates intergroup and class discussions of the investigation and the tasks. Explorations are included to reinforce the knowledge students are expected to have constructed during the first two steps of the cycle.

There is a significant difference between our approach and that of traditional materials: We avoid assigning exercises until *after* there is a reason to believe that students have constructed the relevant knowledge. When one is learning something new, there is a tendency for early interpretations to be inappropriate as students overgeneralize. Working with too many similar examples could cast these inadequacies in stone. With our approach, we hope to reinforce understandings, not *mis*understandings.

Distinguishing Features of the Text

Discovery-Based Learning

Perhaps the most innovative feature of the text is that, very often, students are asked to explore mathematical ideas on the calculator and/or computer before these ideas are discussed in class. These interactive investigations are designed to help students make mathematical discoveries, to support small-group work both in and out of class, and to reduce the teacher's role as lecturer. Although this is a departure from standard practice, both faculty and students who have used the materials feel that this is an important component of our approach to helping students learn mathematical concepts and develop confidence in their ability to direct their own learning. Not only does such a style give students an opportunity to discover mathematical ideas on their own, but even if they don't succeed in making the discovery, the activities form an experiential base that helps them understand an explanation presented in class by another student or by the instructor.

Multiple Representations

This introductory algebra book is based on multiple representations of the function concept. The function concept is central to all topics beginning with Chapter 3; functions are regularly introduced using contextual situations. Students create numeric representations using tables, visual representations using function machines, geometric representations using graphs, and symbolic representations using equations. Students spend time connecting the representation and analyzing which representation might be best for answering a particular question.

Structure

In an effort to develop an appreciation of the underlying structure of algebra, this text explores various ways of thinking about algebra, with the goal of having students exit the course understanding that algebra is much more than "using variables to solve equations." Students are introduced to the idea of structure through the study of a logical system, the S P C (**S**tudent **P**roduction **C**orporation and **S**tart-up, **P**rocess, **C**heck) system. Investigating the S P C system in the first chapter of the book is a clear statement to both students and teachers alike: *This is not the usual introductory algebra book!* Thorough integration of mathematical topics from algebra and number theory helps students see connections and develop understanding and insights not previously experienced and introduces them to the *process* of learning mathematics as well as the *nature of mathematical thinking*.

College-level introductory algebra classes include adult learners with varying mathematical backgrounds and abilities. This text recognizes these college students, respects their life experiences, and places appropriate demands on their cognitive level of development. Because the S P C investigations are placed at the beginning of the text and introduce the concept of variables in a unique way that will be new to all, knowledgeable students and students with weaker algebraic backgrounds start at the same point: prior expertise or lack of it does not affect a student's ability to hypothesize and generate paths using the S P C rules. While investigating the S P C system, they experience the creative side of doing mathematics, learning that "just because a rule exists, doesn't mean you have to use it," "the converse of a rule is not necessarily a rule," and "mathematics is a matter of being consistent." As students discuss and justify their ideas and observe how other students reason, they discover together that mathematics is present even in seemingly non-mathematical situations. Through the S P C system, students also learn to make and test conjectures and acquire an understanding of the importance of notation. Used successfully with middle grade and high school students as well as with students in college developmental mathematics courses, teachers who have piloted the unit containing S P C system report that it has a positive impact on their students' general knowledge of what mathematics is and is not. Students who work through the S P C investigations gain new confidence in their own potential to "do mathematics" that continues throughout the course.

Student Discourse

A conversation between two students, Pete and Sandy, occurs regularly throughout the book. Pete is a student currently enrolled in the course and Sandy is a student who completed the course last term. Their conversations focus on key points of confusion and on making connections among ideas as students proceed through the course.

Glossary

Students are expected to create their own glossary of terms and definitions. A blank Glossary appears at the end of the book. Each set of explorations begins by asking students to write definitions for the new words and phrases encountered in the section. Each new word or phrase is printed in ***boldface italic*** to set it apart.

Reflection A concluding feature of each section is a **Reflection** that asks students to write a short narrative discussing important concepts within a section and between sections.

Concept Map A concluding feature of each chapter is a **Concept Map.** The Concept Maps are centered on some key ideas in the chapter and provide students with a chance to brainstorm the ideas and draw connections between various aspects of the concepts. Detailed instructions and an example appear at the end of Chapter 1.

Student Contribution Another distinctive feature is that the materials include space to write in answers and observations to the investigations in the text. Students develop organizational skills and an ownership in their text as they actually contribute to the writing of the investigation sections during the term.

Unique Chapter Review Each chapter concludes with a review section. In addition to a wide variety of review problems to investigate, students are asked to reflect on the important ideas of the chapter. Suggestions for possible Concept Maps are listed. Finally, for the artistically inclined, an opportunity to draw an illustration of some mathematical person or creature concludes the review section.

Technology Focus

Students use graphing calculators regularly. The graphing calculator is a function machine that has the capability of displaying both input and output on the screen simultaneously so that students can obtain immediate feedback, discover patterns, and identify their previously learned misconceptions. It is recommended (but *not* required) that students have access to a computer algebra system during the course. Investigations done on a computer algebra system allow students to discover symbolic patterns and formulate rules. The use of technology provides opportunities for students to understand and interpret data in terms of the use of tools and techniques that are appropriate to the job. The emphasis is on the development of decision making and problem solving. Students are encouraged to make good choices about the techniques that make the most sense in a particular situation and to check answers for reasonableness and accuracy. We currently use a graphing calculator with a dynamic table feature and a user-friendly software package with symbolic manipulation capability but are not dependent on a specific calculator or on specific computer software.

Skill Development

A key issue in algebra is the development of appropriate skills for later courses. However, we believe that problem-solving skills that encourage independent learning are more important than rote manipulation skills taught out of context. We cover most skills traditionally included in an algebra course, plus advanced skills, because students confront interesting problems

that require these skills. The focus is on interesting problems that require certain skills in the process of investigating the problem. We prefer not to cover skills out of context. We want to emphasize connections among mathematical ideas and skills. This requires a major change in sequencing. If student learning is connected, the chance for retrieval of the knowledge in other problem situations is much greater. Based on constructivist principles, students will develop reasoning from a numerical standpoint by looking for patterns. By generalizing these patterns, they will create mathematics that is their own—this holds true for both concepts and skills.

Instructor Support

A support system must be in place for those who choose to implement materials that are radically different. We were awarded an NSF grant to aid us in developing workshops and an ongoing network for faculty who use this text. Publisher-supported workshops may be available on request. Contact your local Addison Wesley Longman sales consultant for details.

Acknowledgments

There are many people to whom we need to say thank you. We would like to acknowledge specifically Frank Demana and Bert Waits for their pioneering efforts in promoting the widespread acceptance and use of technology to help students explore and visualize mathematics and who served as models for us.

Since the basis for much of the development of this text is grounded in the mathematics education research literature, we would also like to acknowledge the contributions of Jim Kaput and the members of the Algebra Working Group of the National Center for Research in Mathematical Sciences Education. Other researchers who have had a major impact on our work include Jere Confrey, Robert B. Davis, Ed Dubinsky, Jim Fey, Eddie Gray, Carolyn Kieran, Keith Schwingendorf, Richard Skemp, David Tall, Pat Thompson, Zalman Usiskin, and Sigrid Wagner. To this dedicated, articulate group of individuals, and to all those whose research guided our efforts and shaped our vision about what algebra should be, we say thank you.

We also need to acknowledge the major contribution of all those who have field-tested draft versions of this text, along with their students. The insightful recommendations for improvement, editorial comments, and energetic efforts of these pioneer field-testers and students have been invaluable in making the text more useful and student friendly. In addition to the class testers of the preliminary edition listed below, we would like to thank field-testers at Eastern Michigan University and National-Louis University for their early interest in this project.

Preliminary Edition Class Testers

Baker College

Central Washington University

Cerritos College
Chaminade University of Honolulu
City University of New York La Guardia Community College
Clarke College
College of Lake County
College of the Desert
Corning Community College
De Anza College
Edison Community College
El Camino College
Fresno City College
Lakeland Community College
Manchester Community-Technical College
Massachusetts Bay Community College
Naugatuck Valley Community-Technical College
Northwestern Michigan College
Phoenix College
Purdue University Calumet
Richland Community College
Rowan-Cabarrus Community College - North
Siena Heights College
Southeastern Louisiana University
Stephen F. Austin State Univesity
Tennessee Technological University
Texas A & M University
Tomball College
Unity College
University of Hawaii Maui Community College
University of Houston - Downtown
University of Tennessee at Chattanooga
Western Connecticut State University
William Rainey Harper College

A remarkable group of colleagues—Lana Taylor, Gail Johnson, Nancy Rice, and Colette Currie—deserve a special mention. They were willing to test a very early draft of these materials in their classrooms without first seeing a completed project, receiving a chapter at a time during that first semester back in the spring of 1992. There are no words that adequately

acknowledge the gift of trust and support they gave to us in the initial stages of this writing project.

We greatly appreciate the detailed feedback and commentaries provided us by all the reviewers of the various drafts of this text. The reviews addressed specific points of concern, validated our vision of the direction we chose, and helped us clarify content issues.

Reviewers

Rick Armstrong, *St. Louis Community College at Florissant Valley*

Deann Christianson, *University of the Pacific*

Dennis Ebersole, *Northampton County Area Community College*

Grace Foster, *Beaufort County Community College*

Vernon Kays, *Richland Community College*

Joy McMullen, *Lakeland Community College*

Debra Pharo, *Northwestern, Michigan College*

Barbara Sausen, *Freso City College*

Lana Taylor, *Siena Heights College*

Susan White, *DeKalb College*

Tom Williams, *Rowan-Cabarrus Community College*

Finally, we also wish to acknowledge our present editorial staff Jason Jordan, Kari Heen, Michelle Fowler, and Sara Peterson, along with our former editors, Anne Kelly, Greg McRill and Karin Wagner, for their support during the development of this text. Their courage to allow this text to be developed with minimal editorial influence has resulted in what we believe to be truly a reform curriculum. Our production staff, Susan Carsten, Kathy Manley and Joe Vetere should be recognized for their design and technical expertise in preparing this text. We also wish to thank our marketing staff, Andy Fisher, Liz O'Neil and Mark Harrington for their support in the promotion of this text.

We invite you to join the ongoing discussion among colleagues who are using these materials. We encourage you to share your experiences, both successful and not so successful, with others who are attempting to change, not only the content, but their instructional practices as well. We welcome your comments and suggestions for improving this text. Let us know what works and what doesn't so that we can continue to improve these materials. Our e-mail addresses are listed below. Contact us if you would like to become part of the network.

Sincerely,

Phil DeMarois
pdemaroi@harper.cc.il.us

Mercedes McGowen
mmcgowen@harper.cc.il.us

Darlene Whitkanack
darlene155@aol.com

Contents

1

What Is Mathematics?

SECTION 1.1

Learning Mathematics

Purpose

- Investigate the process of thinking about the procedure of doing mathematics (***metacognition***).
- Explore a problem in several ways.

STUDENT DISCOURSE

We would like to introduce you to Pete and Sandy. Pete is a student who is currently taking a course using this book. Sandy is a friend of Pete who took this course last term. Sandy enjoyed the course. She worked hard and did very well. You will find conversations between Pete and Sandy throughout the book that reflect a student's view of the material, including discussions about what is ahead, help to organize what was just covered, and reviews of topics that often cause students problems. Pete and Sandy welcome you as you work through the course. We hope you find their discussions helpful throughout the course.

Pete: I sure don't know what to expect out of this class. Didn't you take the course last semester?

Sandy: Yes. I took the course using these materials and the graphing calculator. It was really different from what I had expected. I had to work hard, but it paid off. I finally feel like I have an idea of how math is useful to me and that I can do it.

Pete: I looked through the text—it sure is different. What is all this investigation and discussion stuff?

Sandy: To really learn math, you have to "do" mathematics yourself. You investigate problems first and write your answers to the investigations right in the book. I found I learned a lot.

Pete: I don't get it! How am I supposed to know the answer if nobody has told me how to do the problem?

Sandy: It's okay if you can't answer an investigation question immediately. Just try to answer each question. You'll go to class with questions about what you don't understand. Trying to write down answers to each investigation helped me find out what I did know and what I didn't. It also helps to talk with classmates before the next class and discuss what you've done and still have questions about.

Pete: Isn't that cheating?

Sandy: No! We were encouraged to work in groups. Three or four heads are better than one. Talk with your classmates and try to put into words what you are struggling with.

Pete: What if I make a mistake?

Sandy: Don't worry if you make a mistake. Making mistakes is part of learning. We were not expected to know all the answers—but we were expected to know where we were having trouble and to ask questions when we came to class.

Pete: How will I know if I am on the right track?

Sandy: After completing an investigation, carefully read the discussion that follows. It deals with the ideas that you explored in the investigation and helps you connect these ideas to mathematics you've already learned. I found that the discussion provided answers to most of my questions before I came to class. It helps if you use a highlighter to mark what you think are the most important ideas.

Pete: Okay. Let me see if I have this right. I'm supposed to investigate problems first and write my answers to the book's questions right in the book. I'm supposed to discuss my answers with other classmates. Reading the discussions that follow will help me see if my thinking is on the right track. And, highlighting what I think is important helps me identify the main ideas.

Sandy: That's basically it. The best way to get the idea is to work on some problems.

Pete: It's pretty weird, but I'll give it a try.

INVESTIGATION

Full Parking Lot: All 20 parking spaces in my favorite parking lot are filled. Some are occupied by motorcycles, and others by cars. Some people count to 10 when they get angry, but that wasn't nearly far enough. I counted wheels—66 to be exact. How many cars and how many motorcycles have invaded my territory?

You can do this problem. It is not important which technique or strategy you use. Follow your intuition and find some way to determine a solution. As you work on this problem, answer the following investigations in the text.

1. Have you ever worked a problem like this before?

2. Describe *in words* what you have done to investigate the problem.

3. What information do you need to know that is not given in the problem statement?

4. What is your solution?

5. Are you sure your solution is correct? Why or why not? How do you normally verify whether you have a correct solution?

6. Now work with a partner. If you were stuck, ask questions and see whether you can use the hints to solve the problem together. Don't just copy your partner's solution. If you both have solutions, see whether your answers are the same. Did you both work the problem using the same strategy?

7. What assumptions did you make about the number of wheels?

DISCUSSION

Questions like the parking lot problem are very interesting to mathematicians because there are several very different strategies that lead to a solution.

The following strategies might have been used on the given problem.

- **Guess and test:** Guess a solution and test whether the answer matches all the conditions.
- **Draw a picture:** Can you visualize the parking lot? You don't have to draw the vehicles, just the wheels.
- **Use logic and arithmetic computation:** Identify the numbers in the problem. Use only numbers given or implied to find the solution.
- **Make an organized list:** Use a table to display information about the problem in an organized way.
- **Write an equation:** Use either the number of cars or the number of motorcycles as a variable to write an equation that represents the problem.
- **Write a system of equations:** Use the number of cars as one variable and the number of motorcycles as a second variable. Write two equations that represent the relationships between the number of cars and the number of motorcycles.

INVESTIGATION

8. Working in groups, try to solve the problem using each of these strategies.

9. Which of the strategies seem more mathematical to you? List your criteria for *mathematical*.

10. Which strategy seems "best"? List your criteria for *best*.

11. Which of the strategies would you like to learn more about?