COMPLEX VARIABLE METHODS
in
ELASTICITY

Complex Variable Methods in Elasticity

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PREFACE

The formulation of the linear theory of elasticity was largely completed by the middle of the nineteenth century. Since this time the theory has been applied to many problems of importance in engineering and found to yield surprisingly good agreement with physical reality. Perhaps the class of problems which has received most attention has been the plane problems of elasticity in view of their direct practical importance and as approximations to three-dimensional situations. This attention has resulted in a great variety of mathematical methods being applied to their solution. By far the most powerful of these methods is the complex variable approach of Kolosov and Muskhelishvili.

The aims of this text are to give a brief description of this method, illustrating the connexion between the most common boundary value problems of two-dimensional elasticity and certain boundary conditions on functions of a complex variable, and finally to describe the techniques of solution of these problems using complex function theory.

Following a brief chapter describing certain results from complex function theory (which may be omitted until required), the basic equations of the plane strain and generalized plane stress problems of elasticity are derived and their solution expressed in terms of functions of a complex variable. The third and fourth chapters illustrate the use of this representation in the solution of elastic boundary value problems for half-planes and circular regions respectively. These chapters may be read independently but the ideas of continuation are developed in Chapter 3 for the algebraically simpler case of the half-plane. The fifth chapter describes the extension of the methods of Chapter 4 to regions which may be conformally mapped onto a circle by rational functions. Each chapter is followed by a number of problems.

Some knowledge of the linear theory of elasticity and an acquaintance with functions of a complex variable would be useful prerequisites for a study of this book.

As it has not been possible to include a comprehensive list of references to the very numerous works in this field those chosen have been selected to illustrate particular points of difficulty or interest arising in the text.

It is a pleasure to record my thanks to my friends at Nottingham, particularly Professor A. J. M. Spencer and Dr W. A. Green, for their interest in this project, to Professor W. Prager of Brown University but for whom this monograph would not have been written, and to Professor E. T. Onat of Yale University for affording me a most stimulating year there during which some of the manuscript was completed. Finally my thanks are due to Mrs E. Burch and Mrs J. C. Lunn for their efficient typing of the manuscript and to my wife for her active encouragement.

Nottingham **A. H. ENGLAND**

INTRODUCTION

Functions of a complex variable were introduced into plane elastic problems in 1909 by Kolosov,[33] who, together with Muskhelishvili,[34] inspired a large school of co-workers in the U.S.S.R.* The resulting developments have been described by Muskhelishvili in two outstanding works which are now available in translation.[43, 44] Other excellent accounts of these techniques have been given by Sokolnikoff,[59] Green and Zerna[21] and Milne-Thomson,[41] and further references may be found in these books and in the survey by Teodorescu.[63]

The method of presentation adopted here is as follows. It is shown that the plane problems of linear elasticity reduce to the solution of Navier's displacement equations of equilibrium subject to certain boundary conditions. On writing the Navier equations in complex variable notation a representation is derived for the elastic displacement in terms of two arbitrary holomorphic functions of a complex variable (the complex potentials). The boundary conditions may then be expressed as certain functional equations relating the complex potentials on the boundaries of the elastic body. Since the solution of these functional equations is extremely difficult for bodies of general shape only the simplest regions will be considered. Attention is concentrated primarily on the half-plane and circular regions for which several methods of solution are available.

The most powerful method of solution employs analytic continuation and is illustrated for half-plane problems in Chapter 3 and for circular regions in Chapter 4. Alternatively, in circular regions, the complex potentials may be represented in terms of either power series or Cauchy integrals along the boundary which enable some boundary value problems to be

* This approach was apparently overlooked elsewhere and a subsequent independent use of the method was made by Stevenson,[61] Green[19] and Milne-Thomson.[40]

solved very conveniently. Particular examples are given in Chapter 4. Finally these methods are applied to the class of bodies which may be conformally mapped onto circular regions.

For reasons of space, certain aspects of the plane problems of elasticity have been omitted. In particular, apart from a few references, there is little description of alternative methods of solution or of extensions of the complex variable methods to deal with strips and wedges or general multiply connected regions. Similarly, plane problems for anisotropic bodies are not considered but accounts of this generalization have been given by Lekhnitskii,[35] Green and Zerna[21] and Milne-Thomson.[41] Finally no mention has been made of the equivalence of the plane problems of elasticity to the bending of plates or to the problems of fluid flow nor is a comparison made between this theory and the plane problems of finite elasticity as discussed by Green and Adkins.[20]

CONTENTS

1

FUNCTIONS OF A COMPLEX VARIABLE

In this chapter some of the basic definitions and properties of functions of a complex variable are stated as a preliminary to their use in later sections. It is hoped that sufficient detail has been included to enable readers to resolve points of difficulty without frequent recourse to the standard texts on this subject.[7,28,43,44]

1.1 Basic definitions

In the following it will be assumed that all definitions refer to curves and regions lying entirely in the complex plane.

An *arc* is a continuous non-intersecting line which has a continuously varying tangent except at a finite number of points. A *contour* is a simple closed arc, for example an ellipse.

We shall refer to an open connected set in the plane as a *region*. When a region which we denote by S^+ has one or more non-intersecting contours as its boundary, the positive sense of description of each contour is taken to be that for which the region S^+ lies to the left. For example when S^+ is bounded internally by the contours $C_1, C_2, \ldots, C_n$ and externally by the contour C_0, then $C_1, C_2, \ldots, C_n$ have a clockwise sense of description and C_0 anticlockwise. This is illustrated in Figure 1.1 for the case $n = 2$. We

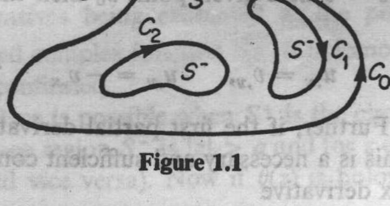

Figure 1.1

denote the open set exterior to S^+ and the bounding contours by S^-, so that on moving in the positive sense along a bounding contour, S^+ lies to the left and S^- to the right. In general S^+ is a multiply connected region, being simply connected only when S^+ is bounded by a single contour C_0.

1.2 Complex functions

Let S be an arbitrary point set in the complex plane, if to each point $z_0 = x_0 + iy_0$ of S there corresponds a complex number $u(x_0, y_0) + iv(x_0, y_0)$ we say that a complex function $\theta(z)$ has been defined on S. The value of the function is

$$\theta(z) = u(x, y) + iv(x, y) \tag{1.1}$$

at the point $z = x + iy$ where u, v are real functions of the variables x, y. We note that a specific functional dependence on z rather than say $\bar{z} = x - iy$ is not assumed by this notation. For example $\theta(z) = \bar{z}$ is a complex function.

In view of the relations

$$z = x + iy, \qquad \bar{z} = x - iy$$
$$x = \tfrac{1}{2}(z + \bar{z}), \qquad y = 1/2i\,(z - \bar{z})$$

let us *define* the operators $\partial/\partial z$, $\partial/\partial \bar{z}$ as for an ordinary coordinate transformation by the relations

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \end{aligned} \tag{1.2}$$

Then

$$\begin{aligned} 2\frac{\partial \theta}{\partial \bar{z}} &= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = u_{,x} - v_{,y} + i(u_{,y} + v_{,x}) \\ 2\frac{\partial \theta}{\partial z} &= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = u_{,x} + v_{,y} - i(u_{,y} - v_{,x}). \end{aligned} \tag{1.3}$$

We see that if $\partial\theta/\partial\bar{z} = 0$ at a given point z_0 then the Cauchy–Riemann equations

$$u_{,x} = v_{,y}, \qquad u_{,y} = -v_{,x} \tag{1.4}$$

are satisfied at z_0. Further, if the first partial derivatives of u and v are continuous at z_0, this is a necessary and sufficient condition for the existence of the complex derivative

$$\frac{d\theta}{dz} = \theta'(z) = \lim_{\delta z \to 0} \frac{\theta(z + \delta z) - \theta(z)}{\delta z}$$

of $\theta(z)$ at the point z_0. In this case it is simple to show from (1.3) that

$$\frac{\partial\theta}{\partial z} = \theta'(z).$$

Definition A function $\theta(z)$ is said to be ***holomorphic***† in a region S^+ if it is single valued in S^+ and its complex derivative $\theta'(z)$ exists at each point of S^+.

For clarity we shall often state when a function is single valued.

1.3 Properties of holomorphic functions

1. If $\theta(z)$ is holomorphic in S^+, then all derivatives of $\theta(z)$ exist and are holomorphic in S^+.

2. If $\theta(z)$ is an arbitrary holomorphic function defined in S^+ then for certain regions S^+ it is possible to use this function to define an associated complex function which is holomorphic in the region which is the image of S^+ in its boundary. This property is of fundamental importance in the method of solution of boundary value problems by continuation. We illustrate this property by defining the associated complex functions for the cases where S^+ is a half plane and a circular region.

Let us denote the half planes $y > 0$ by S^+ and $y < 0$ by S^-. Suppose $\theta(z)$ is holomorphic for $z \in S^+$ then $\theta(\bar{z})$ is defined for all $z \in S^-$ (since for $z \in S^-$, $\bar{z}$ lies in S^+). We now show that the function $\overline{\theta(\bar{z})}$, which is defined for all $z \in S^-$, is holomorphic in S^- and moreover

$$\frac{\mathrm{d}}{\mathrm{d}z}\overline{(\theta(\bar{z}))} = \overline{\theta'(\bar{z})} \quad (z \in S^-). \tag{1.5}$$

From (1.1) and (1.2)

$$\overline{\theta(\bar{z})} = u(x, -y) - \mathrm{i}v(x, -y) \quad (y < 0)$$

and hence

$$\frac{\partial}{\partial \bar{z}}\overline{(\theta(\bar{z}))} = \tfrac{1}{2}\{u_{,x} - v_{,y} - \mathrm{i}(u_{,y} + v_{,x})\} = 0$$

these partial derivatives being evaluated at the point $(x, -y)$, $y < 0$. Hence the associated complex function $\overline{\theta(\bar{z})}$ is holomorphic and by inspection (1.5) may be confirmed.

A similar procedure is possible when S^+ is the circular region $|z| < a$. In this case the image region S^- is $|z| > a$ and for $z \in S^-$ the image point $a^2/\bar{z}$ lies in S^+ (and vice versa). Now if $\theta(z)$ is holomorphic in S^+ then

† The terms analytic and regular are often used.

$\overline{\theta(a^2/\bar{z})}$ is defined for $z \in S^-$ and may be shown to be holomorphic in S^-. In this case however

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\overline{\theta\left(\frac{a^2}{\bar{z}}\right)}\right) = -\frac{a^2}{z^2}\,\overline{\theta'\left(\frac{a^2}{\bar{z}}\right)} \quad (z \in S^-). \tag{1.6}$$

Clearly it is possible to interchange the regions S^+ and S^-. Thus if $\theta(z)$ is holomorphic in S^+ $(|z| > a)$ then the associated function $\overline{\theta(a^2/\bar{z})}$ is holomorphic in S^- $(|z| < a)$ and satisfies (1.6) at all points except $z = 0$ where, it will be seen from (1.6), $\overline{\theta(a^2/\bar{z})}$ has an isolated singularity.

3. *The Continuation Theorem.* Suppose $\theta_1(z)$ and $\theta_2(z)$ are holomorphic functions defined in regions S_1 and S_2. Suppose S_1 and S_2 intersect in a domain S and there exists an infinite sequence of distinct points $\{z_n\}$ in S with at least one limit point in S on which

$$\theta_1(z_n) = \theta_2(z_n) \quad (n = 1, 2, \ldots).$$

Then the function

$$\theta(z)\begin{cases} = \theta_1(z) & (z \in S_1) \\ = \theta_2(z) & (z \in S_2) \end{cases}$$

is holomorphic in the union of S_1 and S_2 and $\theta_2(z)$ is the analytic continuation of $\theta_1(z)$ into S_2, $\theta_1(z)$ the analytic continuation of $\theta_2(z)$ into S_1. It often occurs that S_1 and S_2 intersect in a contour L and that along L

$$\theta_1(z) = \theta_2(z) \quad (z \in L).$$

In this case $\theta(z)$ defined as above is holomorphic in $S_1 + S_2 + L$.

4. A holomorphic function may be represented by a unique uniformly convergent power series of the form $\sum_{n=0}^{\infty} \alpha_n(z - z_0)^n$ in the neighbourhood of any point z_0 in its region of holomorphy.

5. We note that if $\theta(z)$ is holomorphic and single valued in the whole plane including the point at infinity then $\theta(z)$ is a constant. This is Liouville's Theorem.

6. *Laurent's Theorem.* If $\theta(z)$ is holomorphic (and single valued) in the annulus $0 < R_1 < |z - z_0| < R_2 < \infty$ then $\theta(z)$ may be represented by a unique uniformly convergent series (the Laurent Series) $\sum_{n=-\infty}^{\infty} \alpha_n(z - z_0)^n$ in the interior of the annulus.

7. *Cauchy's Theorem.* If $\theta(z)$ is a holomorphic function in the region en-

closed by a contour C and is continuous on C then

$$\int_C \theta(z)\,dz = 0.$$

Note that the region enclosed by a single contour C is simply connected.

1.4 Multiple-valued functions

In this monograph we shall restrict our attention to the multiply connected region S^+ which is bounded internally by the set of contours $C_1, C_2, \ldots, C_n$ and externally by the contour C_0 as shown in Figure 1.1. We now determine a representation for the integral of a function which is holomorphic (and single valued) in S^+. For convenience we denote the holomorphic function by $\theta'(z)$ and its integral by $\theta(z)$. If we choose some fixed point z_0 in S^+ then

$$\theta(z) = \int_L \theta'(z)\,dz$$

where L is some arc lying entirely in S^+ and joining z_0 with the current point z. The possibility now exists that by choosing different arcs L in S^+ different values of $\theta(z)$ result.

Let us suppose first of all that S^+ is simply connected (which corresponds to the absence of the internal boundaries $C_1, \ldots, C_n$, i.e. no holes) then if L_1 and L_2 are different paths joining z_0 and z we find

$$\int_{L_1} \theta'(z)\,dz = \int_{L_2} \theta'(z)\,dz - \int_{L_1 - L_2} \theta'(z)\,dz.$$

However since $\theta'(z)$ is holomorphic and single valued within and on the contour $L_1 - L_2$ Cauchy's Theorem (Section 1.3) implies the latter integral is zero. Consequently $\theta(z)$ is independent of the choice of the arc L and is single valued in any simply connected region.

The general multiply connected region S^+ may be made simply connected by introducing n (non-intersecting) cuts joining each of the internal boundaries $C_1, \ldots, C_n$ to the boundary C_0, see Figure 1.2. We

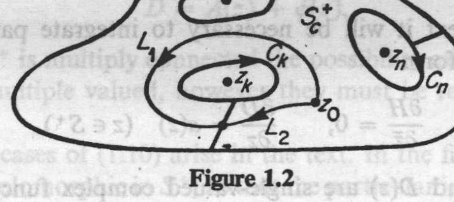

Figure 1.2

denote the cut region by S_C^+. This done it will be seen that $\theta(z)$ is single valued in the cut region S_C^+, but the values of $\theta(z)$ on opposite sides of the cuts will, in general, be different. Consider two arcs L_1 and L_2 lying entirely in S_C^+ which join the fixed point z_0 to corresponding points on opposite sides of the cut between C_k and C_0, see Figure 1.2. Then the change in $\theta(z)$ due to an anticlockwise circuit along the arcs L_1 and L_2 is

$$\int_{L_1 - L_2} \theta'(z)\,\mathrm{d}z.$$

Again we note that $\theta'(z)$ is holomorphic and single valued in S^+ and in particular in the region between the contour $L_2 - L_1$ and C_k, so that by Cauchy's Theorem (Section 1.3),

$$\int_{L_2 - L_1} \theta'(z)\,\mathrm{d}z = -\int_{C_k} \theta'(z)\,\mathrm{d}z = \alpha_k$$

remembering C_k is described clockwise. Thus, for all possible arcs L_1 and L_2 and all points on the cut, $\theta(z)$ increases by a constant α_k in a single anticlockwise circuit of a contour surrounding C_k. Since this type of multi-valuedness holds for each contour C_k $(k = 1, 2, \ldots, n)$, a convenient representation for $\theta(z)$ may be derived.

Consider the function $\log(z - z_k)$ where z_k is a point in the interior of C_k (i.e. outside S^+) then $\log(z - z_k)$ is holomorphic in the cut region S_C^+ and in an anticlockwise circuit around C_k its value increases by $2\pi\mathrm{i}$. Hence the function

$$\theta^*(z) = \theta(z) - \sum_{k=1}^{n} \frac{\alpha_k}{2\pi\mathrm{i}} \log(z - z_k)$$

is continuous across the cuts and so is single valued in S^+. Thus $\theta(z)$ has the representation

$$\theta(z) = \sum_{k=1}^{n} \frac{\alpha_k}{2\pi\mathrm{i}} \log(z - z_k) + \theta^*(z) \tag{1.7}$$

where $\theta^*(z)$ is holomorphic and single valued in the multiply connected region S^+.

Later in the text it will be necessary to integrate partial differential equations of the form

$$\frac{\partial H}{\partial \bar{z}} = 0, \qquad \frac{\partial D}{\partial z} = a(z) \quad (z \in S^+) \tag{1.8}$$

in which $H(z)$ and $D(z)$ are single-valued complex functions in S^+. As some care is required in determining their solutions in the multiply connected region S^+ we examine them in detail here.

Let us write $H(z) = r(x, y) + is(x, y)$ and assume the real functions r and s have single-valued continuous first partial derivatives in S^+. From (1.3), $\partial H/\partial \bar{z} = 0$ implies r and s satisfy the Cauchy–Riemann equations (1.4) and hence the complex derivative $H'(z) = r_{,x} - ir_{,y}$ exists and is single valued in S^+.

If we now assume $H(z)$ is single valued in S^+ we can immediately assert $H(z)$ is holomorphic in S^+. Alternatively, rather than assuming $H(z)$ to be single valued, let us assume its second partial derivatives are continuous and single valued in S^+. In this case it may be confirmed that $H'(z)$ is holomorphic in S^+ and consequently $H(z)$ must be a multiple-valued function in S^+ having a representation of the form (1.7).

The general solution of the homogeneous equation $\partial D/\partial z = 0$ may be derived in a similar manner. On noting that $\partial D/\partial z = 0$ implies $\partial \bar{D}/\partial \bar{z} = 0$ and assuming the first and second partial derivatives of D are continuous and single valued in S^+ we may conclude that $\overline{D(z)}$ is a multiple-valued function of the form (1.7). Thus, under these assumptions, the equation $\partial D/\partial z = 0$ has the solution $D = \overline{\phi(z)}$ where

$$\phi(z) = \sum_{k=1}^{n} \frac{\beta_k}{2\pi i} \log(z - z_k) + \phi^*(z) \tag{1.9}$$

and $\phi^*(z)$ is holomorphic in S^+.

Let us now consider the more general equation

$$\frac{\partial D}{\partial z} = a(z) \cdot \quad (z \in S^+) \tag{1.10}$$

and assume that D and its first and second partial derivatives are continuous and single valued in S^+. To be consistent the complex function $a(z)$ must be assumed single valued in S^+. As $\partial/\partial z$ is a linear operator it will be seen that D is the sum of a particular integral of (1.10), say $D = A(z)$, and a general solution of the homogeneous equation $\partial D/\partial z = 0$ as derived above. Thus the general solution of (1.10) is

$$D = A(z) + \overline{\phi(z)}.$$

Note that as S^+ is multiply connected the possibility exists that both $A(z)$ and $\phi(z)$ are multiple valued, however they must be related so that D is single valued.

Two special cases of (1.10) arise in the text. In the first $\partial D/\partial z = \theta'(z)$, where $\theta'(z)$ is holomorphic in S^+, so that the particular integral is $D = \theta(z)$

(from (1.7)) and the general solution is

$$D = \theta(z) + \overline{\phi(z)}. \tag{1.11}$$

As both $\theta(z)$ and $\phi(z)$ are multiple valued, having representations of the form (1.7) and (1.9), D will be seen to increase by the constant $\alpha_k + \bar{\beta}_k$ in a single anticlockwise circuit of any contour in S^+ surrounding only C_k. Hence as D is required to be single valued the general solution is (1.11) where $\theta(z)$ is given by (1.7) and $\phi(z)$ by (1.9) in which $\beta_k = -\bar{\alpha}_k$ $(k = 1, 2, \ldots, n)$.

In the second case $\partial D/\partial z = \overline{\theta'(z)}$ where $\theta'(z)$ is holomorphic in S^+. Now the particular integral is seen to be $D = z\,\overline{\theta'(z)}$ and is single valued in S^+. Consequently, as D is single valued, the general solution is

$$D = z\,\overline{\theta'(z)} + \overline{\phi(z)} \tag{1.12}$$

where $\phi(z)$ is holomorphic in S^+.

As an illustration of the above points and as a particular example of the theory of Section 2.9 we determine the solution of Laplace's equation

$$\nabla_1^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = 0$$

in the region S^+ where u is a real single-valued function of x and y with continuous second partial derivatives. On using the definitions (1.2) we see that

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) = 0 \quad \text{in } S^+.$$

Hence $\partial u/\partial z$ must be a function which is holomorphic in S^+ which, for convenience, we denote by

$$\frac{\partial u}{\partial z} = \theta'(z).$$

Now from (1.11) the solution of this equation is $u = \theta(z) + \overline{\phi(z)}$ where $\phi(z)$ is an arbitrary function of the form (1.9). Further, as u is real, we must conclude

$$u = \theta(z) + \overline{\theta(z)}.$$

Finally since $\theta(z)$ must have the form (1.7)

$$u = \sum_{k=1}^{n} \frac{1}{2\pi i}\{\alpha_k \log(z - z_k) - \bar{\alpha}_k \log(\bar{z} - \bar{z}_k)\} + \theta^*(z) + \overline{\theta^*(z)}$$

and is single valued in S^+ only if $\alpha_k + \bar{\alpha}_k = 0$ for $k = 1, 2, \ldots, n$. Hence