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Modular Interfaces

Modular Lie Algebras, Quantum Groups, and Lie Superalgebras

Vyjayanthi Chari and Ivan B. Penkov, Editors

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Volume 4

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A Conference in Honor
of Richard E. Block
February 18–20, 1995
University of California, Riverside

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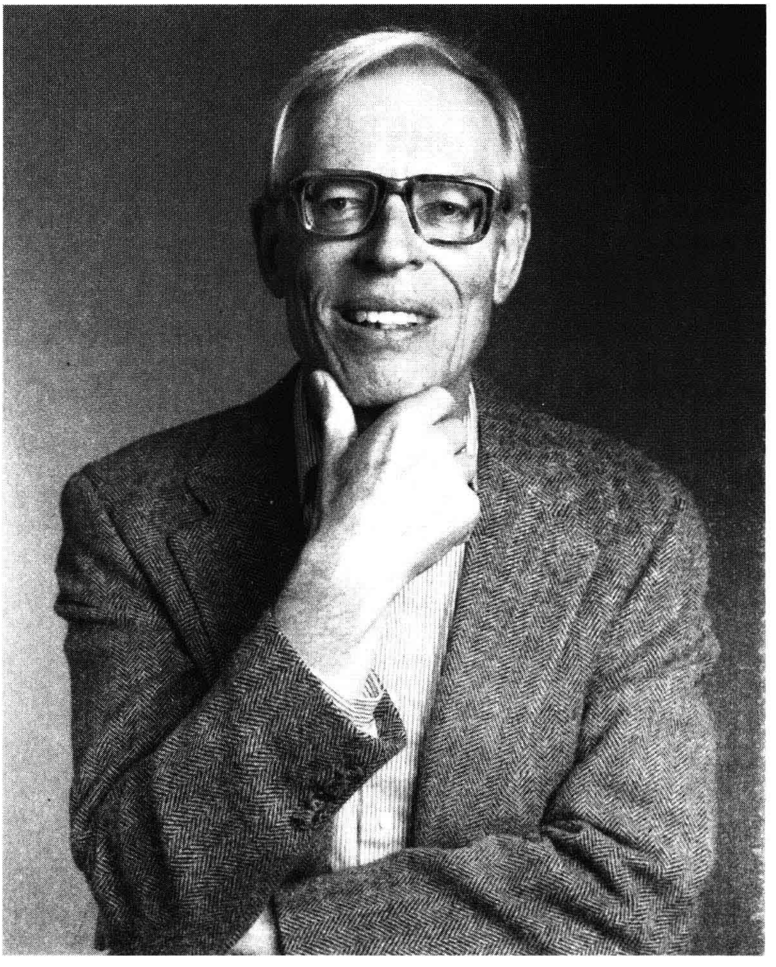

Richard E. Block

Modular Interfaces

Preface

This is a collection of papers dedicated to Richard E. Block and is the Proceedings of a conference held at the University of California, Riverside, February 18–20, 1995 on the occassion of his retirement. The conference was titled Modular Interfaces and focussed on the interplay between the theory of Lie algebras of prime characteristic, quantum groups and Lie superalgebras.

Professor Block's research has largely been devoted to the study of Lie algebras in characteristic p: in particular the classification at prime characteristic of simple Lie algebras. The outstanding achievement in this direction is the result that he proved jointly with Robert Wilson, that for $p > 7$, the restricted simple Lie algebras are of classical or Cartan type. In proving this they established the Kostrikin–Shafarevic conjecture.

Professor Block has also made major contributions to other areas of Lie theory and algebra. We mention just two of them here since they help to explain the topics of the conference: his work on differentiably simple rings which was used (by just replacing p–truncated polynomials with the exterior algebra) to give the structure of semisimple Lie superalgebras of characteristic zero, and his work on Hopf algebras. The study of Lie superalgebras has been important for some time now and has ramifications in physics as well. As for Hopf algebras, these have attracted a great deal of attention, since M. Jimbo and V.G. Drinfeld independently defined quantized enveloping algebras, or quantum groups, which are deformations of Lie algebras in the category of Hopf algebras. Further, G.Lusztig discovered that there is a remarkable connection between the representation theory of quantum groups associated to simple Lie algebras at a root of unity and representations of modular Lie algebras.

It thus seemed that a conference on this subject in honor of Richard Block was particularly appropriate. The conference proved to be exciting with very stimulating talks. We are indebted to the many participants (some of whom had to find their own funds), both from the United States and abroad: their presence was invaluable in making the conference successful.

The organizing committee consisted of Robert E. Blattner, Vyjayanthi Chari, Gary Griffing, Ivan B. Penkov and Robert L. Wilson. We acknowledge financial assistance from the NSF. We were also supported from a variety of university sources: the Department of Mathematics, from University funds provided to Professor O. Viro as the holder of the F. Burton Jones Chair, and the Dean's Office at UCR.

The editors wish to thank many people associated with the conference and with the Proceedings. We are grateful to Professor S.T. Yau for his generosity in offering us the services of International Press and to Professor V.S. Varadarajan, whose gentle persuasion had a good deal to do with getting this volume together and published.

We thank the chair of the department, Albert Stralka, whose help was crucial in getting the conference organized. The staff in the Mathematics office in particular, Linda Terry, Danielle McQueen and Susan Spranger were extremely efficient in taking care of many of the practical details involved in the organization of the conference and we are grateful for their help. We thank Jan Patterson for her assistance in getting the articles ready for these proceedings. Finally, we thank our graduate students Benjamin Edwards and Ivan Dimitrov for their help with all kinds of practical matters during the conference.

Vyjayanthi Chari

Ivan B. Penkov

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Highest Weight Modules for Locally Finite Lie Algebras

YURI BAHTURIN AND GEORGIA BENKART

Introduction

There have been a number of important investigations in the representation theory of various classes of infinite-dimensional Lie algebras which generalize the finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero. However, the general representation theory of one natural class of such algebras, namely the locally finite Lie algebras, remains virtually unexplored. These algebras arise in conjunction with the structure theory of infinite-dimensional Lie algebras and deformations of their universal enveloping algebras and in the study of identical relations in groups and Lie algebras, for example, in the study of Burnside-type problems. They also provide a parallel notion to locally finite groups, whose theory is well-developed (see [**KW**]).

An algebra L is *locally finite-dimensional*, or *locally finite* for short, if each finite subset of L generates a finite-dimensional subalgebra. The Lie algebra $gl(\infty)$ of infinite matrices which have only finitely many nonzero entries is a well-studied example (see [**K, §7.11**]). The algebra $gl(\infty)$ may be regarded as the direct limit of the general linear Lie algebras $gl(n)$ of $n \times n$ matrices, and indeed, that is the appropriate viewpoint to adopt when dealing with locally finite Lie algebras. The recent papers of the first author and H. Strade [**BS1-BS3**] exhibit a striking assortment of locally finite simple Lie algebras over fields of arbitrary characteristic which arise as direct limits of finite-dimensional simple Lie algebras. All the algebras considered in [**BS1-BS3**] fall into the general framework of Lie algebras of infinite matrices with only a finite number of nonzero diagonals. Such Lie algebras are discussed in [**KR**], and the monograph of Kac and Raina describes many applications of locally finite simple Lie algebras such as $gl(\infty)$ to various problems in mathematical physics. The infinite rank affine algebras, A_∞, $A_{+\infty}$, B_∞, C_∞, D_∞, (see [**K, §7.11**]) are also locally finite algebras, and their representations find applications in Olshanskiĭ's work [**O1-O3**] on Yangians (certain deformations of the universal enveloping algebras of loop algebras). Using results of Enright, Howe, and Wallach [**EHW**] and the Jantzen form, Natarajan [**N**] has studied unitarity questions for highest weight modules of direct limits of simple Lie algebras coming from corner embeddings (see Sec. 3 below). All these prior investigations have

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focused on very specific (and in general very well-behaved) classes of locally finite algebras.

Rather than reinvent fundamental results for each particular variety of locally finite algebras, in this present paper we initiate research into the general representation theory of locally finite Lie algebras. We restrict ourselves from the outset to modules with local systems of submodules to make the problems more tractable and the results more applicable. One of our main concerns is with the behavior of the weight lattice of locally finite Lie algebras. This is of special interest because for many types of locally finite Lie algebras roots do not exist. As we show, in many instances weights do not exist either. But for certain types of embeddings that respect the triangular decompositions, weights do exist, and thus highest weight modules arise naturally too in this context.

It is premature at this stage to make far-ranging predictions about the future role of this subject. However, the representation theory of locally finite Lie algebras leads naturally to very classical problems about the embedding of one finite-dimensional simple Lie algebra into another which preserves their triangular decompositions, to the corresponding mapping of the weight lattices induced by such an embedding, and so forth. Some of these questions are touched upon in this article, but many more are omitted. In particular, questions about the characters of highest weight modules over locally finite algebras, about the tensor products of such modules, etc. are left for subsequent investigations.

1. Direct Limits of Modules

1.1 Let I be a directed partially ordered set (poset) indexing a collection $\{U^{(i)}\}_{i\in I}$ of rings (or algebras over a field F). Then for every $i_1, i_2 \in I$ there exists $i_3 \in I$ such that $i_1, i_2 \leq i_3$. Suppose for each $i \in I$ there corresponds a nonempty directed poset $J^{(i)}$ such that each $j \in J^{(i)}$ labels a left $U^{(i)}$-module $M^{(j)}$. When $j \in J^{(i)}$, set $\overline{j} = i$. Assume the disjoint union $J = \cup_{i\in I} J^{(i)}$ is a directed poset with the property that $j_1 \leq j_2$ for $j_1 \in J^{(i_1)}$ and $j_2 \in J^{(i_2)}$ implies $\overline{j_1} = i_1 \leq i_2 = \overline{j_2}$.

1.2 Now suppose further that whenever $i_1, i_2 \in I$ satisfy $i_1 \leq i_2$ there is a ring homomorphism $\phi_{i_2,i_1} : U^{(i_1)} \longrightarrow U^{(i_2)}$ such that $\phi_{i_3,i_2}\phi_{i_2,i_1} = \phi_{i_3,i_1}$ whenever $i_1 \leq i_2 \leq i_3$ and $\phi_{i,i} = 1_{U^{(i)}}$. Similarly assume that when $j_1 \leq j_2$ there is a homomorphism $\psi_{j_2,j_1} : M^{(j_1)} \longrightarrow M^{(j_2)}$ of abelian groups (or of vector spaces if we are working with algebras and their modules over some field F) such that $\psi_{j_3,j_2}\psi_{j_2,j_1} = \psi_{j_3,j_1}$ whenever $j_1 \leq j_2 \leq j_3$ and $\psi_{j,j} = 1_{M^{(j)}}$. These compatibility conditions allow us to construct the direct limit $\mathcal{U} = \varinjlim U^{(i)}$ and the direct limit $\mathcal{M} = \varinjlim M^{(j)}$. We say that $(\mathcal{U}, \mathcal{M}) = (U^{(i)}, M^{(j)})_{i\in I, j\in J}$ is a *direct limit of modules* if for any $j_1, j_2 \in J$ with $j_1 \leq j_2$ the following diagram is commutative:

$$\begin{array}{ccc} U^{(\overline{j_1})} \otimes M^{(j_1)} & \xrightarrow{\mu_{j_1}} & M^{(j_1)} \\ \phi_{\overline{j_2},\overline{j_1}} \otimes \psi_{j_2,j_1} \downarrow & & \downarrow \psi_{j_2,j_1} \\ U^{(\overline{j_2})} \otimes M^{(j_2)} & \xrightarrow{\mu_{j_2}} & M^{(j_2)} \end{array} \tag{1.3}$$

Here $\mu_j : U^{(\overline{j})} \otimes M^{(j)} \longrightarrow M^{(j)}$ denotes the mapping $\mu_j : u \otimes m \longrightarrow u \cdot m$ giving the module structure on $M^{(j)}$.

1.4 When $u \in U^{(i)}$ and $x \in M^{(j)}$, choose $k \in J$ with $\overline{k} = i$ and $\ell \in J$ with $j, k \leq \ell$. Then $i \leq \overline{\ell}$, and we define

$$u * x = \phi_{\overline{\ell},i}(u) \cdot \psi_{\ell,j}(x). \tag{1.5}$$

PROPOSITION 1.6. *Suppose that $(\mathcal{U}, \mathcal{M})$ is a direct limit of modules. Then (1.5) defines a $\mathcal{U}$-module structure on $\mathcal{M}$ such that $M^{(j)}$ is a $U^{(\overline{j})}$-submodule of $\mathcal{M}$.*

PROOF. We have to show that if u is replaced by an equivalent $u' \in U^{(i')}$, x by $x' \in M^{(j')}$, k by k' with $\overline{k'} = i'$, and ℓ by ℓ' with $j', k' \leq \ell'$, then

$$w' = \phi_{\overline{\ell'},i'}(u') \cdot \psi_{\ell',j'}(x')$$

is equivalent to the right side of (1.5), which we denote by w. However, since u' is equivalent to u in $\mathcal{U}$, there is an $m \in I$ with $i, i' \leq m$ such that $\phi_{m,i}(u) = \phi_{m,i'}(u')$. Likewise, there exists $n \in J$ with $j, j' \leq n$ such that $\psi_{n,j}(x) = \psi_{n,j'}(x')$. Assume $r \in J$ is such that $\overline{r} = m$. If $s \in J$ is chosen greater than ℓ, ℓ', n, r, then $i, i' \leq \overline{r} \leq \overline{s}$, and the following holds:

$$\begin{aligned} \psi_{s,\ell}(w) &= \psi_{s,\ell}(\phi_{\overline{\ell},i}(u) \cdot \psi_{\ell,j}(x)) = \phi_{\overline{s},\overline{\ell}}(\phi_{\overline{\ell},i}(u)) \cdot \psi_{s,\ell}(\psi_{\ell,j}(x)) \\ &= \phi_{\overline{s},i}(u) \cdot \psi_{s,j}(x) = \phi_{\overline{s},m}(\phi_{m,i}(u)) \cdot \psi_{s,n}(\psi_{n,j}(x)) \\ &= \phi_{\overline{s},m}(\phi_{m,i'}(u')) \cdot \psi_{s,n}(\psi_{n,j'}(x')) \\ &= \phi_{\overline{s},i'}(u') \cdot \psi_{s,j'}(x') \\ &= \phi_{\overline{s},\overline{\ell'}}(\phi_{\overline{\ell'},i'}(u')) \cdot \psi_{s,\ell'}(\psi_{\ell',j'}(x')) \\ &= \psi_{s,\ell'}(\phi_{\overline{\ell'},i'}(u') \cdot \psi_{\ell',j'}(x')) \\ &= \psi_{s,\ell'}(w'). \end{aligned}$$

Thus, the action in (1.5) is well-defined. If $x \in M^{(j)}$ and $u \in U^{(\overline{j})}$, then since (1.5) is independent of the choice of ℓ, we may take it to equal j. As a result, $u * x = \phi_{\overline{j},\overline{j}}(u) \cdot \psi_{j,j}(x) = u \cdot x$, so that $M^{(j)}$ is indeed a $U^{(\overline{j})}$-submodule with its original structure. Suppose we want to verify that a particular module axiom holds - for example, $(u_1 u_2) * x = u_1 * (u_2 * x)$ for elements $u_1 \in U^{(i_1)}, u_2 \in U^{(i_2)}, x \in M^{(j)}$. Take $k_1, k_2 \in J$ with $\overline{k_1} = i_1$, $\overline{k_2} = i_2$, and $\ell \in J$ with $j, k_1, k_2 \leq \ell$. Then

$$\begin{aligned} (u_1 u_2) * x &= \left(\phi_{\overline{\ell},i_1}(u_1)\phi_{\overline{\ell},i_2}(u_2)\right) \cdot \psi_{\ell,j}(x) \\ &= \phi_{\overline{\ell},i_1}(u_1) \cdot \left(\phi_{\overline{\ell},i_2}(u_2) \cdot \psi_{\ell,j}(x)\right) \\ &= u_1 * (u_2 * x). \end{aligned}$$

Arguing in this way we see that $\mathcal{M}$ is an $\mathcal{U}$-module. $\square$

1.7 A generic example of the above is a module $\mathcal{M}$ over a unital ring $\mathcal{U}$ where $\mathcal{M}$ and $\mathcal{U}$ have *local systems of submodules and subrings* respectively. By this we mean that $\mathcal{U}$ has a set $I = \{S\}$ of proper subrings S such that

(i) $\mathcal{U} = \cup_{S\in I} S$, and

(ii) for any $R, S \in I$ there exists $T \in I$ such that $R, S \subseteq T$.

Associated to each $S \in I$ is a collection J_S of proper S-submodules of $\mathcal{M}$, and for the disjoint union $J = \cup_{S\in I} J_S$ of the sets J_S, it is assumed that the following hold:

(iii) $\mathcal{M} = \cup_{M\in J} M$, and

(iv) for any $L \in J_R$, $M \in J_S$, there is a $T \in I$ and a $N \in J_T$ such that $R, S \subseteq T$ and $L, M \subseteq N$.

The set I is ordered by inclusion: $R \leq S$ if $R \subseteq S$. If $M \in J_R$, $N \in J_S$, the ordering on J is specified by setting $M \leq N$ if $R \subseteq S$ and $M \subseteq N$. The mappings $\phi_{S,R} : R \longrightarrow S$ and $\psi_{N,M} : M \longrightarrow N$ are the natural inclusions. To verify that (1.3) holds, suppose that $M \in J_R$, and for $R \leq S$ assume $N \in J_S$ satisfies $M \subseteq N$. Then for $u \in R$ and $x \in M$,

$$\begin{aligned}\psi_{N,M} \circ \mu_M(u \otimes x) &= \psi_{N,M}(u \cdot x) = u \cdot x \\ &= \phi_{S,R}(u) \cdot \psi_{N,M}(x) = \mu_N\Big(\phi_{S,R}(u) \otimes \psi_{N,M}(x)\Big) \\ &= \mu_N \circ (\phi_{S,R} \otimes \psi_{N,M})(u \otimes x),\end{aligned}$$

as required. Thus, $(\mathcal{U}, \mathcal{M}) = (S, M)_{S\in I, M\in J}$ is a direct limit of modules.

The next result provides a criterion for the irreducibility of a module having a local system of submodules:

PROPOSITION 1.8. *Assume $\mathcal{M}$ is a module over a ring $\mathcal{U}$ with a local system $I = \{S\}$ of subrings and a local system $J = \cup_{S\in I} J_S$ of submodules. Then $\mathcal{M}$ is irreducible if and only if given any R-submodule $M \in J_R$ and any proper submodule $M' \subset M$, there exists an $S \in I$ with $R \subset S$ and an $N \in J_S$ with $M \subset N$ such that $M' \neq M \cap N'$ for any S-submodule N' of N.*

PROOF. Assume the condition is violated by an R-submodule M' of M for some $R \in I$ and $M \in J_R$. For any $S \supset R$ and $N \in J_S$ with $N \supset M$, there exists an S-submodule $N' \subset N$ with $M' = M \cap N'$. Then for $\widetilde{N} \stackrel{\text{def}}{=} S * M'$, $\widetilde{N} \subseteq N'$. We claim that $\mathcal{N} = \cup_{M\leq N\in J} \widetilde{N}$ is a proper $\mathcal{U}$-submodule of $\mathcal{M}$. Given any $N \in J_S$ with $N \supseteq M$ and any subring T of $\mathcal{U}$ there is a subring $U \supseteq S, T$ and a submodule $P \in J_U$ so that $P \supseteq N \supset M$. Then $\widetilde{P} \stackrel{\text{def}}{=} U * M' = U * S * M' = U * \widetilde{N} \supseteq T * \widetilde{N}$ to show that $\mathcal{N}$ is a submodule. Moreover, $M \cap \mathcal{N} \subset \cup_{M\leq N}(M \cap \widetilde{N}) \subset \cup_{M\leq N}(M \cap N') = M'$. As M' is proper, $\mathcal{N} \neq \mathcal{M}$, and since $\mathcal{N}$ contains M' it is nonzero. It follows that $\mathcal{M}$ is not irreducible.

Conversely, suppose that the condition is satisfied, and let $\mathcal{N}$ be a proper submodule of $\mathcal{M}$. Then for some $M \in J$, $M' \stackrel{\text{def}}{=} \mathcal{N} \cap M$ is a proper submodule of M. If

$N \in J$ is such that $M \leq N$, then $(\mathcal{N} \cap N) \cap M = \mathcal{N} \cap M = M'$. Thus, it is possible to find a submodule $N' = \mathcal{N} \cap N$ of N with $N' \cap M = M'$ for each $M \leq N$. This contradiction shows that $\mathcal{M}$ must be irreducible. □

1.9 If each module $M \in J$ is irreducible, then the condition in Proposition 1.8 vacuously holds, and $\mathcal{M}$ is irreducible.

2. Modules with a Unique Maximal Submodule and Verma Modules

THEOREM 2.1. *Let $\mathcal{U}$ be a unital ring and $\mathcal{M}$ be a left $\mathcal{U}$-module with a finite generating set X. Suppose $\mathcal{U}$ has a local system $I = \{S\}$ of subrings and $\mathcal{M}$ a local system $J = \cup_{S \in I}\{M^{(S)}\}$ of submodules where $M^{(S)} = S \cdot X$ for all $S \in I$. If each $M^{(S)}$ has a unique maximal submodule, then so does M.*

PROOF. It suffices to verify that the sum of proper $\mathcal{U}$-submodules is always proper. Since $\mathcal{M}$ is finitely generated it is enough to show that any finite sum of proper submodules is proper, and for this, it is sufficient to handle the case of two proper submodules. If N_1 and N_2 are proper $\mathcal{U}$-submodules, then there exist $S, R \in I$ such that $N_1 \cap M^{(R)}$ and $N_2 \cap M^{(S)}$ are proper in $M^{(R)}$ and $M^{(S)}$ respectively. If $R, S \leq T$ for $T \in I$, then $N_1 \cap M^{(T)}$ and $N_2 \cap M^{(T)}$ are proper in $M^{(T)}$. Since $M^{(T)}$ has a unique maximal submodule, $N_1 \cap M^{(T)} + N_2 \cap M^{(T)}$ is proper in $M^{(T)}$. Now if $N_1 + N_2 = \mathcal{M}$, then $N_1 + N_2 \supset X$, and each $x \in X$ can be written as $x = n_1(x) + n_2(x)$ where $n_i(x) \in N_i$, $i = 1, 2$. There exist $A, B \in I$ with $n_1(x) \in M^{(A)}$ and $n_2(x) \in M^{(B)}$. Choose $C \in I$ with $T, A, B \leq C$. Then $n_i(x) \in M^{(C)} \cap N_i$ for $i = 1, 2$, and $x \in M^{(C)} \cap N_1 + M^{(C)} \cap N_2$, which is a proper submodule of $M^{(C)}$ since $M^{(C)}$ has a unique maximal submodule. Such a $C \in I$ can be found for each $x \in X$, and since X is finite, there is some $D \in I$ such that $X \subset M^{(D)} \cap N_1 + M^{(D)} \cap N_2$. The right-hand side is a proper D-submodule of $M^{(D)}$, but $D \cdot X = M^{(D)}$, a contradiction. □

2.2 Suppose now that F is an algebraically closed field of characteristic zero. Assume the Lie algebra $\mathfrak{g}$ is the direct limit of a family $\{\mathfrak{g}^{(i)} \mid i \in I\}$ (I a directed poset) of finite-dimensional semisimple Lie algebras over F each of which has a triangular decomposition

$$\mathfrak{g}^{(i)} = \mathfrak{n}_-^{(i)} \oplus \mathfrak{h}^{(i)} \oplus \mathfrak{n}_+^{(i)}$$

with respect to some Cartan subalgebra $\mathfrak{h}^{(i)}$ and some choice of base $\Delta^{(i)}$ for the root system $R^{(i)}$ corresponding to $\mathfrak{h}^{(i)}$. We say that the structural homomorphisms $\phi_{j,i} : \mathfrak{g}^{(i)} \longrightarrow \mathfrak{g}^{(j)}$ are *triangular* if they preserve the given triangular decompositions so that $\phi_{j,i}(\mathfrak{n}_\pm^{(i)}) \subseteq \mathfrak{n}_\pm^{(j)}$ and $\phi_{j,i}(\mathfrak{h}^{(i)}) \subseteq \mathfrak{h}^{(j)}$ for all $i, j \in I$ with $i \leq j$. The homomorphism $\phi_{j,i}$ extends to associative algebra homomorphism $\phi_{j,i} : U^{(i)} \longrightarrow U^{(j)}$ of the respective universal enveloping algebras $U^{(i)} = U(\mathfrak{g}^{(i)})$ and $U^{(j)} = U(\mathfrak{g}^{(j)})$. It is clear that

$$U(\varinjlim \mathfrak{g}^{(i)}) = U(\mathfrak{g}) = \varinjlim U^{(i)}.$$

The categories of $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules are equivalent.

2.3 Suppose $\mathfrak{h} = \varinjlim \mathfrak{h}^{(i)}$ and $\mathfrak{n}_\pm = \varinjlim \mathfrak{n}_\pm^{(i)}$. A $\mathfrak{g}$-module M is a *highest-weight module of highest weight* $\lambda \in \mathfrak{h}^*$ if there is a vector $m^+ \in M$ (a *maximal vector*) such that $\mathfrak{n}_+ m^+ = 0$, $U(\mathfrak{g})m^+ = M$, and $h \cdot m^+ = \lambda(h)m^+$ for all $h \in \mathfrak{h}$. The most important example of a highest-weight module is the *Verma module*,

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} Fv^+, \tag{2.4}$$

obtained by inducing from a one-dimensional submodule Fv^+ for the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+ v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$ for all $h \in \mathfrak{h}$. An arbitrary highest-weight $\mathfrak{g}$-module $M = U(\mathfrak{g})m^+$ of highest weight λ is a quotient of $V(\lambda)$ via the map $V(\lambda) \longrightarrow M$ with $u \otimes v^+ \longrightarrow u \cdot m^+$. Moreover, M has a local system of submodules $M^{(i)} = U^{(i)}m^+$, and each $M^{(i)}$ is a highest-weight $\mathfrak{g}^{(i)}$-module of highest weight $\lambda^{(i)} = \lambda \mid_{\mathfrak{h}^{(i)}}$. According to [**H, Theorem 20.2 (d)**] any highest-weight module for $\mathfrak{g}^{(i)}$, such as $M^{(i)}$, has a unique maximal submodule. Thus, by Theorem 2.1, M has a unique maximal submodule. In particular, the Verma module $V(\lambda)$ has a unique maximal submodule, and as a consequence we have the following result.

THEOREM 2.5. *Suppose* $\mathfrak{g} = \varinjlim \mathfrak{g}^{(i)}$ *where the algebras* $\mathfrak{g}^{(i)}$ *are finite-dimensional semisimple Lie algebras over an algebraically closed field of characteristic zero, and assume the structural homomorphisms are triangular. Then* $\mathfrak{g}$ *has a triangular decomposition* $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

(i) A highest-weight $\mathfrak{g}$*-module has a unique maximal submodule.*

(ii) All irreducible highest-weight $\mathfrak{g}$*-modules of highest weight* $\lambda \in \mathfrak{h}^*$ *are isomorphic.*

2.6 It is worth noting that the conventional proof to show that a highest-weight module M of highest weight λ has a unique maximal submodule cannot be applied in general to direct limit algebras. The typical argument (for example, when $\mathfrak{g}$ is a finite-dimensional semisimple algebra or a symmetrizable Kac-Moody algebra) uses the fact that the algebra $\mathfrak{g}$ decomposes into root spaces relative to $\mathfrak{h}$, and hence M into weight spaces, and any proper submodule is contained in the sum of the weight spaces M_μ with $\mu \neq \lambda$. That guarantees that the sum of all proper submodules is proper, and so it is the unique maximal submodule. As the examples at the end of Section 6 illustrate, the algebra $\mathfrak{g} = \varinjlim \mathfrak{g}^{(i)}$ need not have a root space decomposition relative to $\mathfrak{h}$. The argument above circumvents this difficulty. The direct limit algebras $\mathfrak{g}$ which have been studied previously in [**K**], [**KR**], [**N**], and [**O1-O3**] all have root space decompositions and so do not encounter this problem.

2.7 When the conditions of Theorem 2.5 are met, then up to isomorphism there is a unique irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$ which we denote by $L(\lambda)$. Suppose $M = U(\mathfrak{g})m^+$ is some highest-weight $\mathfrak{g}$-module with highest weight λ and let Q be its unique maximal submodule (which exists by Theorem 2.5), so that $L(\lambda) \cong M/Q$. Consider the local system $M^{(i)} = U^{(i)}m^+$ where $U^{(i)} = U(\mathfrak{g}^{(i)})$. Each $M^{(i)}$ is a highest-weight module of highest weight $\lambda^{(i)} = \lambda \mid_{\mathfrak{h}^{(i)}}$ and so has a unique maximal submodule $N^{(i)}$. Now $Q = \cup_{i\in I} Q^{(i)}$ where $Q^{(i)} = Q \cap M^{(i)}$. Since $Q^{(i)}$ is a proper $\mathfrak{g}^{(i)}$-submodule of $M^{(i)}$, we have $Q^{(i)} \subseteq N^{(i)}$. If the submodule N generated by the $N^{(i)}$ is proper, then $N = Q$ and $M/Q = M/N$ has a local system $\{M^{(i)}/N^{(i)}\}$ of modules which are irreducible highest-weight modules with highest weight $\lambda^{(i)} = \lambda \mid_{\mathfrak{h}^{(i)}}$. In this case we may say $L(\lambda) = \underrightarrow{\lim} L(\lambda^{(i)})$.

3. Corner and Triangular Embeddings

3.1 To illustrate triangular embeddings of the kind discussed in the last section, we present next certain embeddings of split semisimple Lie algebras. First consider the infinite matrices over F in

$$gl(\infty) = \{a = (a_{i,j})_{i,j\in\mathbb{Z}} \mid a_{i,j} = 0 \text{ if } |i|, |j| >> 0\} \tag{3.2}$$

with the Lie bracket given by the matrix commutator $[a,b] = ab - ba$. Then the standard matrix units $e_{i,j}, i,j \in \mathbb{Z}$, constitute a basis for $gl(\infty)$. The span of the matrix units $e_{i,j}$ where $-n \leq i,j \leq n$ determines a general linear Lie algebra $gl(2n+1)$, and $\text{span}_F\{e_{i,j} \mid -n \leq i,j \leq n-1\}$ forms a copy of $gl(2n)$. Clearly $gl(m) \subset gl(m')$ if $m < m'$ and $gl(\infty) = \underrightarrow{\lim} gl(m)$. The subalgebra

$$A_\infty \stackrel{\text{def}}{=} \{a \in gl(\infty) \mid \text{tr}(a) = 0\} \tag{3.3}$$

is the direct limit $\underrightarrow{\lim} sl(m)$ of the special linear subalgebras $sl(m) = \{a \in gl(m) \mid \text{tr}(a) = 0\}$.

3.4 Suppose $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices in $\mathfrak{g} = A_\infty$, and assume $\epsilon_i : \mathfrak{h} \longrightarrow F$ is the projection of a matrix onto the (i,i) entry. Then the matrix unit $e_{i,j}$ with $i \neq j$ corresponds to the root $\epsilon_i - \epsilon_j$ relative to $\mathfrak{h}$, and the set of roots $\Delta = \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}\}$ determines a base of the root system Φ of $\mathfrak{g}$. Relative to this choice of base, the positive roots are $\Phi^+ = \{\epsilon_i - \epsilon_j \mid -\infty < i < j < \infty\}$, and the algebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$ (resp. $\mathfrak{n}_- = \sum_{\alpha<0} \mathfrak{g}_\alpha$) is the space of upper (resp. lower) triangular matrices. Each subalgebra $sl(m)$ has an analogous triangular decomposition into lower triangular, diagonal, and upper triangular matrices, and the natural inclusion $sl(m) \longrightarrow sl(m')$ for $m < m'$ is a triangular embedding.

3.5 The spaces $gl(+\infty) = \text{span}_F\{e_{i,j} \mid 1 \leq i,j < \infty\}$ and $A_{+\infty} = \{a \in gl(+\infty) \mid \text{tr}(a) = 0\}$ are subalgebras of $gl(\infty)$. Moreover, $gl(+\infty) = \underrightarrow{\lim} gl(m)$ where $gl(m) = \text{span}_F\{e_{i,j} \mid 1 \leq i,j \leq m\}$, and $A_{+\infty} = \underrightarrow{\lim} sl(m)$ where $sl(m) = \{a \in gl(m) \mid \text{tr}(a) = 0\}$. Relative to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, $sl(m)$ has a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ into lower triangular, diagonal, and upper triangular matrices, and the embeddings $sl(m) \longrightarrow sl(m')$ for $m < m'$ are also triangular.

3.6 For $k = 1, \ldots, m$ we define the m-complement of k to be

$$\overline{k} = m + 1 - k \tag{3.7}$$

and set

$$(3.8) \qquad \zeta_k = \begin{cases} 1 & \text{if } k = 1, \dots, n \\ -1 & \text{if } k = n+1, \dots, m = 2n \end{cases} \qquad \text{(C)}$$

$$\zeta_k = 1 \qquad \text{(B), (D)}.$$

3.9 If $y_{i,j} = \zeta_i e_{i,j} - \zeta_j e_{\bar{j},\bar{i}}$, and

$$(3.10) \qquad \begin{aligned} B_n &= \text{span}_F\{y_{i,j} \mid 1 \leq i, j \leq m = 2n+1\} \\ C_n &= \text{span}_F\{y_{i,j} \mid 1 \leq i, j \leq m = 2n\} \\ D_n &= \text{span}_F\{y_{i,j} \mid 1 \leq i, j \leq m = 2n\}, \end{aligned}$$

where the ζ's are as in (3.8)(B),(D) for B_n and D_n and (3.8)(C) for C_n, then with respect to the Lie bracket

$$(3.11) \qquad [y_{i,j}, y_{k,\ell}] = \delta_{j,k}\zeta_j y_{i,\ell} - \delta_{i,\ell}\zeta_i y_{k,j} + \delta_{\bar{j},\ell}\zeta_j y_{k,\bar{i}} - \delta_{\bar{i},k}\zeta_i y_{\bar{j},\ell},$$

which comes from the Lie bracket on $gl(\infty)$, these are split simple algebras of types B_n,C_n and D_n (with the exception $D_2 \cong A_1 \oplus A_1$).

3.12 Suppose when considering $sl(n)$ or $gl(n)$ that $y_{i,j} = e_{i,j}$ for $1 \leq i, j \leq n$. The algebra $sl(n)$ is a split simple Lie algebra of type A_{n-1} whenever the characteristic of F is zero or p and p doesn't divide n. When p divides n, there is a one-dimensional center spanned by the identity element modulo which it is simple. Suppose $Y = A, B, C, D$, and let Y_n denote the corresponding algebra. Then there is a natural embedding of Y_n into Y_{n+1} taking $y_{i,j}$ to $y'_{i,j}$ where the $y'_{i,j}$'s are the spanning elements of Y_{n+1}. We refer to these embeddings as *corner embeddings*. In each algebra $\text{span}_F\{y_{i,i}\} \cap Y_n$ determines a Cartan subalgebra consisting of diagonal matrices. The $y_{i,j}$'s with $i < j$ are upper triangular matrices and the $y_{i,j}$'s with $i > j$ are lower triangular matrices. The corner embeddings are triangular with respect to the decomposition into lower triangular, diagonal, and upper triangular matrices. The resulting algebras $Y_{+\infty} = \varinjlim Y_n$ for $Y = A, B, C, D$ thus have triangular decompositions. The natural inclusion of Y_n, $Y = B, C, D$, into $sl(m)$ or $gl(m)$ ($m = 2n+1$ or $2n$), is also a triangular embedding.

3.13 The elements

$$h_i = y_{i,i} - y_{i+1,i+1} \qquad \text{for } i = 1, \dots, n-1,$$

$$h_n = \begin{cases} 2y_{n,n} & (B) \\ y_{n,n} & (C) \\ y_{n-1,n-1} + y_{n,n} & (D) \end{cases},$$

span the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices in the algebra of type A_{n-1}, B_n, C_n