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Yevhen G. Zelenyuk

ULTRAFILTERS AND TOPOLOGIES ON GROUPS

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Preface

This book presents the relationship between ultrafilters and topologies on groups. It shows how ultrafilters are used in constructing topologies on groups with extremal properties and how topologies on groups serve in deriving algebraic results about ultrafilters.

The contents of the book fall naturally into three parts. The first, comprising Chapters 1 through 5, introduces to topological groups and ultrafilters insofar as the semigroup operation on ultrafilters is not required. Constructions of some important topological groups are given. In particular, that of an extremally disconnected topological group based on a Ramsey ultrafilter. Also one shows that every infinite group admits a nondiscrete zero-dimensional topology in which all translations and the inversion are continuous.

In the second part, Chapters 6 through 9, the Stone–Čech compactification βG of a discrete group G is studied. For this, a special technique based on the concepts of a local left group and a local homomorphism is developed. One proves that if G is a countable torsion free group, then βG contains no nontrivial finite groups. Also the ideal structure of βG is investigated. In particular, one shows that for every infinite Abelian group G, βG contains $2^{2^{|G|}}$ minimal right ideals.

In the third part, using the semigroup βG, almost maximal topological and left topological groups are constructed and their ultrafilter semigroups are examined. Projectives in the category of finite semigroups are characterized. Also one shows that every infinite Abelian group with finitely many elements of order 2 is absolutely ω-resolvable, and consequently, can be partitioned into ω subsets such that every coset modulo infinite subgroup meets each subset of the partition.

The book concludes with a list of open problems in the field.

Some familiarity with set theory, algebra and topology is presupposed. But in general, the book is almost self-contained. It is aimed at graduate students and researchers working in topological algebra and adjacent areas.

Johannesburg, November 2010 Yevhen Zelenyuk

Contents

Chapter 1
Topological Groups

In this chapter some basic concepts and results about topological groups are presented. The largest group topology in which a given filter converges to the identity is described. As an application Markov's Criterion of topologizability of a countable group is derived. Another application is computing the minimum character of a nondiscrete group topology on a countable group which cannot be refined to a nondiscrete metrizable group topology. We conclude by proving Arnautov's Theorem on topologizability of a countably infinite ring.

1.1 The Notion of a Topological Group

Definition 1.1. A group G endowed with a topology is a *topological group* if the multiplication

$$\mu : G \times G \ni (x, y) \mapsto xy \in G$$

and the inversion

$$\iota : G \ni x \mapsto x^{-1} \in G$$

are continuous mappings. A topology which makes a group into a topological group is called a *group topology*.

The continuity of the multiplication and the inversion is equivalent to the continuity of the function

$$\mu' : G \times G \ni (x, y) \mapsto xy^{-1} \in G.$$

Indeed, $\mu'(x, y) = \mu(x, \iota(y))$, $\iota(x) = \mu'(1, x)$ and $\mu(x, y) = \mu'(x, \iota(y))$.

The continuity of μ' means that whenever $a, b \in G$ and U is a neighborhood of ab, there are neighborhoods V and W of a and b, respectively, such that

$$VW^{-1} \subseteq U.$$

It follows that whenever $a_1, \dots, a_n \in G$, $k_1, \dots, k_n \in \mathbb{Z}$ and U is a neighborhood of $a_1^{k_1} \dots a_n^{k_n} \in G$, there are neighborhoods $V_1, \dots, V_n$ of $a_1, \dots, a_n$, respectively, such that

$$V_1^{k_1} \cdots V_n^{k_n} \subseteq U.$$

Another immediate property of a topological group G is that the translations and the inversion of G are homeomorphisms. Indeed, for each $a \in G$, the *left translation*

$$\lambda_a : G \ni x \mapsto ax \in G$$

and the *right translation*

$$\rho_a : G \ni x \mapsto xa \in G$$

are continuous mappings, being restrictions of the multiplication. The inversion ι is continuous by the definition. Since we have also that $(\lambda_a)^{-1} = \lambda_{a^{-1}}$, $(\rho_a)^{-1} = \rho_{a^{-1}}$ and $\iota^{-1} = \iota$, all of them are homeomorphisms.

A topological space X is called *homogeneous* if for every $a, b \in X$, there is a homeomorphism $f : X \to X$ such that $f(a) = b$. If G is a topological group and $a, b \in G$, then $\lambda_{ba^{-1}} : G \to G$ is a homeomorphism and $\lambda_{ba^{-1}}(a) = ba^{-1}a = b$. Thus, we have that

Lemma 1.2. *The space of a topological group is homogeneous.*

Now we establish some separation properties of topological groups.

Lemma 1.3. *Every topological group satisfying the T_0 separation axiom is regular and hence Hausdorff.*

In this book, by a *regular* space one means a T_3-space.

Proof. Let G be a T_0 topological group. We first show that G is a T_1-space. Since G is homogeneous, it suffices to show that for every $x \in G \setminus \{1\}$, there is a neighborhood U of 1 not containing x. By T_0, there is an open set U containing exactly one of two points $1, x$. If $1 \in U$, we are done. Otherwise xU^{-1} is a neighborhood of 1 not containing x.

Now we show that for every neighborhood U of 1, there is a closed neighborhood of 1 contained in U. Choose a neighborhood V of 1 such that $VV^{-1} \subseteq U$. Then for every $x \in G \setminus U$, one has $xV \cap V = \emptyset$. Indeed, otherwise $xa = b$ for some $a, b \in V$, which gives us that $x = ba^{-1} \in VV^{-1} \subseteq U$, a contradiction. Hence $\operatorname{cl} V \subseteq U$. □

In fact, the following stronger statement holds.

Theorem 1.4. *Every Hausdorff topological group is completely regular.*

Proof. See [55, Theorem 10]. □

Theorem 1.4 is the best possible general separation result. However, for countable topological groups, it can be improved.

A space is *zero-dimensional* if it has a base of clopen (= both closed and open) sets. Note that if a T_0-space is zero-dimensional, then it is completely regular.

Proposition 1.5. *Every countable regular space is normal and zero-dimensional.*

Proof. Let X be a countable regular space and let A and B be disjoint closed subsets of X. Enumerate A and B as

$$A = \{a_n : n < \omega\} \quad \text{and} \quad B = \{b_n : n < \omega\}.$$

Inductively, for each $n < \omega$, choose neighborhoods U_n and V_n of a_n and b_n respectively such that

(a) $\operatorname{cl} U_n \cap B = \emptyset$ and $A \cap \operatorname{cl} V_n = \emptyset$,

(b) $U_n \cap (\bigcup_{i<n} V_i) = \emptyset$ and $(\bigcup_{i<n} U_i) \cap V_n = \emptyset$, and

(c) $U_n \cap V_n = \emptyset$.

(a) is needed to satisfy (b). Conjunction of (b) and (c) is equivalent to

$$\Big(\bigcup_{i \leq n} U_i\Big) \cap \Big(\bigcup_{i \leq n} V_i\Big) = \emptyset.$$

It follows that

$$U = \bigcup_{n<\omega} U_n \quad \text{and} \quad V = \bigcup_{n<\omega} V_n$$

are disjoint neighborhoods of A and B, respectively.

Now, having established that X is normal, let U be an open neighborhood of a point $x \in X$. Without loss of generality one may suppose that $U \neq X$. Then by Urysohn's Lemma, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus U) = \{1\}$. Since X is countable, there is $r \in [0, 1] \setminus f(X)$. Then $f^{-1}([0, r)) = f^{-1}([0, r])$ is a clopen neighborhood of x contained in U. □

It follows from Lemma 1.3 and Proposition 1.5 that

Corollary 1.6. *Every countable Hausdorff topological group is normal and zero-dimensional.*

Note that every first countable Hausdorff topological group is also normal. (A space is *first countable* if every point has a countable neighborhood base.) This is immediate from the fact that every metric space is normal and the following result.

Theorem 1.7. *A Hausdorff topological group is metrizable if and only if it is first countable. In this case, the metric can be taken to be left invariant.*

Proof. See [34, Theorem 8.3]. □

Starting from Chapter 5, all topological groups are assumed to be Hausdorff.

1.2 The Neighborhood Filter of the Identity

For every set X,

$$\mathcal{P}(X) = \{Y : Y \subseteq X\}.$$

Definition 1.8. Let X be a nonempty set. A *filter* on X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ with the following properties:

(1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,

(2) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and

(3) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

In other words, a filter on X is a nonempty family of nonempty subsets of X closed under finite intersections and supersets. A classic example of a filter is the set $\mathcal{N}_x$ of all neighborhoods of a point x in a topological space X called the *neighborhood filter* of x. By a neighborhood of x one means any set whose interior contains x. The system $\{\mathcal{N}_x : x \in X\}$ of all neighborhood filters on X is called the *neighborhood system* of X.

Theorem 1.9. *Let X be a space and let $\{\mathcal{N}_x : x \in X\}$ be the neighborhood system of X. Then*

(i) *for every $x \in X$ and $U \in \mathcal{N}_x$, $x \in U$, and*

(ii) *for every $x \in X$ and $U \in \mathcal{N}_x$, $\{y \in X : U \in \mathcal{N}_y\} \in \mathcal{N}_x$.*

Conversely, given a set X and a system $\{\mathcal{N}_x : x \in X\}$ of filters on X satisfying conditions (i)–(ii), *there is a unique topology $\mathcal{T}$ on X for which $\{\mathcal{N}_x : x \in X\}$ is the neighborhood system.*

Proof. That the neighborhood system $\{\mathcal{N}_x : x \in X\}$ of a space X satisfies (i)–(ii) is obvious. We need to prove the converse.

Define the operator int on the subsets of X by putting for every $A \subseteq X$

$$\text{int } A = \{x \in X : A \in \mathcal{N}_x\}.$$

We claim that it satisfies the following conditions:

(a) $\text{int } X = X$,

(b) $\text{int } A \subseteq A$,

(c) $\text{int }(\text{int } A) = \text{int } A$, and

(d) $\text{int }(A \cap B) = (\text{int } A) \cap (\text{int } B)$.

Indeed, for every $x \in X$, $X \in \mathcal{N}_x$, consequently $x \in \text{int } X$, and so (a) is satisfied.

For (b), if $x \in \text{int } A$, then $A \in \mathcal{N}_x$, and so by (i), $x \in A$.

To check (c), let $x \in \text{int } A$. Then $A \in \mathcal{N}_x$. Applying (ii) we obtain that $\text{int } A \in \mathcal{N}_x$. It follows that $x \in \text{int}(\text{int } A)$. Hence $\text{int } A \subseteq \text{int}(\text{int } A)$. The converse inclusion follows from (b).

To check (d), let $x \in (\text{int } A) \cap (\text{int } B)$. Then $A \in \mathcal{N}_x$ and $B \in \mathcal{N}_x$, so $A \cap B \in \mathcal{N}_x$. It follows that $x \in \text{int}(A \cap B)$. Hence $(\text{int } A) \cap (\text{int } B) \subseteq \text{int}(A \cap B)$. Conversely, let $x \in \text{int}(A \cap B)$. Then $A \cap B \in \mathcal{N}_x$, consequently $A \in \mathcal{N}_x$ and $B \in \mathcal{N}_x$. It follows that $x \in (\text{int } A) \cap (\text{int } B)$. Hence $\text{int}(A \cap B) \subseteq (\text{int } A) \cap (\text{int } B)$.

It follows from (a)–(d) that there is a unique topology $\mathcal{T}$ on X such that int is the interior operator for $(X, \mathcal{T})$. We have that a subset $U \subseteq X$ is a neighborhood of a point $x \in X$ in $\mathcal{T}$ if and only if $x \in \text{int } U$, and so if and only if $U \in \mathcal{N}_x$. Hence, $\{\mathcal{N}_x : x \in X\}$ is the neighborhood system for $(X, \mathcal{T})$. □

In a topological group, the neighborhood system is completely determined by the neighborhood filter of the identity.

Lemma 1.10. *Let G be a topological group and let $\mathcal{N}$ be the neighborhood filter of 1. Then for every $a \in G$, $a\mathcal{N} = \mathcal{N}a$ is the neighborhood filter of a.*

Here,

$$a\mathcal{N} = \{aB : B \in \mathcal{N}\} \quad \text{and} \quad \mathcal{N}a = \{Ba : B \in \mathcal{N}\}.$$

Proof. Since both λ_a and ρ_a are homeomorphisms and $\lambda_a(1) = \rho_a(1) = a$,

$$a\mathcal{N} = \lambda_a(\mathcal{N}) = \rho_a(\mathcal{N}) = \mathcal{N}a$$

is the neighborhood filter of a. □

The next theorem characterizes the neighborhood filter of the identity of a topological group.

Theorem 1.11. *Let $(G, \mathcal{T})$ be a topological group and let $\mathcal{N}$ be the neighborhood filter of 1. Then*

(1) *for every $U \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $VV \subseteq U$,*

(2) *for every $U \in \mathcal{N}$, $U^{-1} \in \mathcal{N}$, and*

(3) *for every $U \in \mathcal{N}$ and $x \in G$, $xUx^{-1} \in \mathcal{N}$.*

Conversely, given a group G and a filter $\mathcal{N}$ on G satisfying conditions (1)–(3), there is a unique group topology $\mathcal{T}$ on G for which $\mathcal{N}$ is the neighborhood filter of 1. The topology $\mathcal{T}$ is Hausdorff if and only if

(4) $\bigcap \mathcal{N} = \{1\}$.

Note that conditions (2) and (3) in Theorem 1.11 are equivalent, respectively, to

(2′) $\mathcal{N}^{-1} = \mathcal{N}$, and

(3′) for every $x \in G$, $x\mathcal{N}x^{-1} = \mathcal{N}$,

where $\mathcal{N}^{-1} = \{A^{-1} : A \in \mathcal{N}\}$ and $x\mathcal{N}x^{-1} = \{xAx^{-1} : A \in \mathcal{N}\}$.

Proof. That the neighborhood filter of 1 satisfies (1)–(3) follows from the continuity of the multiplication $\mu(x, y)$ at $(1, 1)$ and the mappings $\iota(x)$ and $\lambda_x(\rho_{x^{-1}}(y))$ at 1. To prove the converse, consider the system $\{x\mathcal{N} : x \in G\}$. We claim that it satisfies the conditions of Theorem 1.9.

To check (i), let $x \in G$ and $U \in \mathcal{N}$. It follows from (1)–(2) that there is $V \in \mathcal{N}$ such that $VV^{-1} \subseteq U$. Then $x \in xVV^{-1} \subseteq xU$.

To check (ii), let $x \in G$ and $U \in \mathcal{N}$. It follows from (1) that there is $V \in \mathcal{N}$ such that $VV \subseteq U$. For every $y \in xV$, $yV \subseteq xVV \subseteq xU$, consequently

$$xV \subseteq \{y \in G : xU \in y\mathcal{N}\},$$

and so

$$\{y \in G : xU \in y\mathcal{N}\} \in x\mathcal{N}.$$

Now by Theorem 1.9, there is a unique topology $\mathcal{T}$ on G such that for each $x \in G$, $x\mathcal{N}$ is the neighborhood filter of x, that is, the neighborhoods of x are of the form xU, where U is a neighborhood of 1. To see that $\mathcal{T}$ is a group topology, let $a, b \in G$ be given and let U be a neighborhood of 1. Using conditions (1)–(3) choose a neighborhood V of 1 such that $bVV^{-1}b^{-1} \subseteq U$. Then

$$aV(bV)^{-1} = aVV^{-1}b^{-1} = ab^{-1}bVV^{-1}b^{-1} \subseteq ab^{-1}U.$$

Since $\mathcal{T}$ is a group topology, it is Hausdorff if and only if it is a T_1-topology, and so if and only if $\bigcap \mathcal{N} = \{1\}$. □

The notion of a filter is closely related to that of a filter base.

Definition 1.12. Let X be a nonempty set. A *filter base* on X is a nonempty family $\mathcal{B} \subseteq \mathcal{P}(X)$ with the following properties:

(1) $\emptyset \notin \mathcal{B}$, and

(2) for every $A, B \in \mathcal{B}$ there is $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

Equivalently, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a filter base if

$$\mathcal{F} = \{A \subseteq X : A \supseteq B \text{ for some } B \in \mathcal{B}\}$$

is a filter, and in this case we say that $\mathcal{B}$ is a *base* for $\mathcal{F}$. Note that if $\mathcal{F}$ is a filter, then $\mathcal{B} \subseteq \mathcal{F}$ is a base for $\mathcal{F}$ if and only if for every $A \in \mathcal{F}$ there is $B \in \mathcal{B}$ such that $B \subseteq A$.

If X is a topological space and $x \in X$, then a base for the neighborhood filter of x is called a *neighborhood base* at x.

As a consequence we obtain from Theorem 1.11 the following.

Corollary 1.13. *Let $\mathcal{B}$ be a filter base on G satisfying the following conditions:*

(1) *for every $U \in \mathcal{B}$, there is $V \in \mathcal{B}$ such that $VV \subseteq U$,*

(2) *for every $U \in \mathcal{B}$, $U^{-1} \in \mathcal{B}$, and*

(3) *for every $U \in \mathcal{B}$ and $x \in G$, $xUx^{-1} \in \mathcal{B}$.*

Then there is a unique group topology $\mathcal{T}$ on G for which $\mathcal{B}$ is a neighborhood base at 1. *The topology $\mathcal{T}$ is Hausdorff if and only if*

(4) $\bigcap \mathcal{B} = \{1\}$.

1.3 The Topology $\mathcal{T}(\mathcal{F})$

Definition 1.14. For every filter $\mathcal{F}$ on a group G, let $\mathcal{T}(\mathcal{F})$ denote the largest group topology on G in which $\mathcal{F}$ converges to 1.

Definition 1.14 is justified by the fact that for every family $\{\mathcal{T}_i : i \in I\}$ of group topologies on G, the least upper bound $\bigvee_{i\in I} \mathcal{T}_i$ taken in the lattice of all topologies on G is a group topology.

Definition 1.15. For every filter $\mathcal{F}$ on a group G, let $\tilde{\mathcal{F}}$ denote the filter with a base consisting of subsets of the form

$$\bigcup_{x\in G} x(A_x \cup A_x^{-1} \cup \{1\})x^{-1},$$

where for each $x \in G$, $A_x \in \mathcal{F}$.

Lemma 1.16. *For every filter $\mathcal{F}$ on a group G, $\tilde{\mathcal{F}}$ is the largest filter contained in $\mathcal{F}$ such that*

(i) $1 \in \bigcap \tilde{\mathcal{F}}$,

(ii) $\tilde{\mathcal{F}}^{-1} = \tilde{\mathcal{F}}$, *and*

(iii) *for every $x \in G$, $x\tilde{\mathcal{F}}x^{-1} = \tilde{\mathcal{F}}$.*

Proof. That $\tilde{\mathcal{F}}$ satisfies (i) is obvious. To check (ii) and (iii), let $A_x \in \mathcal{F}$ for each $x \in G$. Then

$$\Big(\bigcup_{x\in G} x(A_x \cup A_x^{-1} \cup \{1\})x^{-1}\Big)^{-1} = \bigcup_{x\in G} x(A_x \cup A_x^{-1} \cup \{1\})x^{-1}.$$

Consequently, $\tilde{\mathcal{F}}^{-1} = \tilde{\mathcal{F}}$. Next, for every $y \in G$,

$$\begin{aligned} y\Big(\bigcup_{x\in G} x(A_x \cup A_x^{-1} \cup \{1\})x^{-1}\Big)y^{-1} &= \bigcup_{x\in G} yx(A_x \cup A_x^{-1} \cup \{1\})(yx)^{-1} \\ &= \bigcup_{x\in G} x(A_{y^{-1}x} \cup A_{y^{-1}x}^{-1} \cup \{1\})x^{-1} \\ &= \bigcup_{x\in G} x(B_x \cup B_x^{-1} \cup \{1\})x^{-1}, \end{aligned}$$

where $B_x = A_{y^{-1}x}$ for each $x \in G$. It follows that $y\tilde{\mathcal{F}}y^{-1} = \tilde{\mathcal{F}}$.

To see that $\tilde{\mathcal{F}}$ is the largest filter on G contained in $\mathcal{F}$ and satisfying (i)–(iii), let $\mathcal{G}$ be any such filter and let $A \in \mathcal{G}$. Then $1 \in A$ and for each $x \in G$, there is $A_x \in \mathcal{G}$ such that $x(A_x \cup A_x^{-1})x^{-1} \subseteq A$. Since $\mathcal{G} \subseteq \mathcal{F}$, $A_x \in \mathcal{F}$ for each $x \in G$. Define $B \in \tilde{\mathcal{F}}$ by

$$B = \bigcup_{x\in G} x(A_x \cup A_x^{-1} \cup \{1\})x^{-1}.$$

Then $B \subseteq A$, and so $A \in \tilde{\mathcal{F}}$. □

For every $n \in \mathbb{N}$, let S_n denote the group of all permutations on $\{1, \dots, n\}$. The next theorem describes the topology $\mathcal{T}(\mathcal{F})$.

Theorem 1.17. *For every filter $\mathcal{F}$ on a group G, the neighborhood filter of 1 in $\mathcal{T}(\mathcal{F})$ has a base consisting of subsets of the form*

$$\bigcup_{n=1}^{\infty} \bigcup_{\pi\in S_n} \prod_{i=1}^{n} B_{\pi(i)},$$

where $(B_n)_{n=1}^{\infty}$ is a sequence of members of $\tilde{\mathcal{F}}$.

Proof. It is clear that these subsets form a filter base on G. In order to show that this is the neighborhood filter of 1 in a group topology, it suffices to check conditions (1)–(3) of Corollary 1.13. Let $(B_n)_{n=1}^{\infty}$ be any sequence of members of $\tilde{\mathcal{F}}$.

To check (1), define the sequence $(C_n)_{n=1}^{\infty}$ in $\tilde{\mathcal{F}}$ by $C_n = B_{2n} \cap B_{2n-1}$. Then for for every $n \in \mathbb{N}$ and $\pi, \rho \in S_n$,

$$\prod_{i=1}^{n} C_{\pi(i)} \prod_{i=1}^{n} C_{\rho(i)} \subseteq \prod_{i=1}^{n} B_{2\pi(i)-1} \prod_{i=1}^{n} B_{2\rho(i)} = \prod_{j=1}^{2n} B_{\sigma(j)}$$

where $\sigma \in S_{2n}$ is defined by

$$\sigma(j) = \begin{cases} 2\pi(j) - 1 & \text{if } j \le n \\ 2\rho(j-n) & \text{if } j > n. \end{cases}$$

It follows that

$$\Big(\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}C_{\pi(i)}\Big)\Big(\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}C_{\pi(i)}\Big)\subseteq\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}B_{\pi(i)}.$$

To check (2), define the sequence $(C_n)_{n=1}^{\infty}$ in $\tilde{\mathcal{F}}$ by $C_n = B_n^{-1}$ (Lemma 1.16). Then for every $n\in\mathbb{N}$ and $\pi\in S_n$,

$$\Big(\prod_{i=1}^{n}B_{\pi(i)}\Big)^{-1}=\prod_{i=1}^{n}B_{\rho(i)}^{-1}=\prod_{i=1}^{n}C_{\rho(i)}$$

where $\rho\in S_n$ is defined by $\rho(i)=\pi(n+1-i)$. Consequently,

$$\Big(\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}B_{\pi(i)}\Big)^{-1}=\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}C_{\pi(i)}.$$

To check (3), let $x\in G$. Define the sequence $(C_n)_{n=1}^{\infty}$ in $\tilde{\mathcal{F}}$ by $C_n = xB_nx^{-1}$ (Lemma 1.16). Then for every $n\in\mathbb{N}$ and $\pi\in S_n$,

$$x\Big(\prod_{i=1}^{n}B_{\pi(i)}\Big)x^{-1}=\prod_{i=1}^{n}xB_{\pi(i)}x^{-1}=\prod_{i=1}^{n}C_{\pi(i)}.$$

Consequently,

$$x\Big(\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}B_{\pi(i)}\Big)x^{-1}=\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}C_{\pi(i)}.$$

Now let G be endowed with any group topology in which $\mathcal{F}$ converges to 1 and let U be a neighborhood of 1. Note that every neighborhood of 1 is a member of $\tilde{\mathcal{F}}$ (Lemma 1.16). Choose inductively a sequence $(V_n)_{n=0}^{\infty}$ of neighborhoods of 1 such that $V_0 = U$ and for every n,

$$V_{n+1}V_{n+1}V_{n+1}\subseteq V_n.$$

Then whenever $n_1,\dots,n_k$ are distinct numbers in $\mathbb{N}$, one has

$$V_{n_1}\cdots V_{n_k}\subseteq V_n,$$

where $n=\min\{n_1,\dots,n_k\}-1$. (To see this, pick $i\in\{1,\dots,k\}$ such that $n_i=\min\{n_1,\dots,n_k\}$ and write $V_{n_1}\cdots V_{n_k}$ as $(V_{n_1}\cdots V_{n_{i-1}})V_{n_i}(V_{n_{i+1}}\cdots V_{n_k})$.) It follows that

$$\bigcup_{n=1}^{\infty}\bigcup_{\pi\in S_n}\prod_{i=1}^{n}V_{\pi(i)}\subseteq U,$$

and so U is a neighborhood of 1 in $\mathcal{T}(\mathcal{F})$. □

1.4 Topologizing a Group

Definition 1.18. Let G be a countably infinite group. Enumerate G as $\{g_n : n < \omega\}$ without repetitions and with $g_0 = 1$.

(i) For every infinite sequence $(a_n)_{n=1}^\infty$ in G, define $U((a_n)_{n=1}^\infty) \subseteq G$ by

$$U((a_n)_{n=1}^\infty) = \bigcup_{n=1}^{\infty} \bigcup_{\pi \in S_n} \prod_{i=1}^{n} B_{\pi(i)},$$

where $B_i = \bigcup_{j=0}^\infty g_j\{1, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, \dots\} g_j^{-1}$.

(ii) For every finite sequence $a_1, \dots, a_n$ in G, define $U(a_1, \dots, a_n) \subseteq G$ by

$$U(a_1, \dots, a_n) = \bigcup_{\pi \in S_n} \prod_{i=1}^{n} B_{\pi(i)}^n,$$

where $B_i^n = \bigcup_{j=0}^{n-i} g_j\{1, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, \dots, a_n^{\pm 1}\} g_j^{-1}$. That is, $U(a_1, \dots, a_n)$ consists of all elements of the form

$$g_{j_1} c_1 g_{j_1}^{-1} \cdots g_{j_n} c_n g_{j_n}^{-1},$$

where $j_i \in \{0, \dots, n - \pi(i)\}$ and $c_i \in \{1, a_{\pi(i)+j_i}^{\pm 1}, \dots, a_n^{\pm 1}\}$ for each $i = 1, \dots, n$, and $\pi \in S_n$. In particular, $U(a_1) = \{1, a_1^{\pm 1}\}$. Also put $U(\emptyset) = \{1\}$.

(iii) For every finite sequence $a_1, \dots, a_{n-1}$ in G, let $T(a_1, \dots, a_{n-1}, x)$ denote the set of group words $f(x)$ in the alphabet $G \cup \{x\}$ in which variable x occurs and which have the form

$$f(x) = g_{j_1} c_1 g_{j_1}^{-1} \cdots g_{j_n} c_n g_{j_n}^{-1},$$

where $j_i \in \{0, \dots, n - \pi(i)\}$ and $c_i \in \{1, a_{\pi(i)+j_i}^{\pm 1}, \dots, a_{n-1}^{\pm 1}, x^{\pm 1}\}$ for each $i = 1, \dots, n$, and $\pi \in S_n$. In particular, $T(x)$ consists of two group words x and x^{-1}.

Of course, in the case where G is Abelian, all these definitions look simpler. In particular,

$$B_i^n = \{0, \pm a_i, \dots, \pm a_n\} \quad \text{and} \quad U(a_1, \dots, a_n) = \sum_{i=1}^{n} B_i^n.$$

Theorem 1.19. *For every sequence $(a_n)_{n=1}^\infty$ in G, the following statements hold:*

(1) *$U((a_n)_{n=1}^\infty)$ is a neighborhood of 1 in $\mathcal{T}((a_n)_{n=1}^\infty)$,*

(2) *$U((a_n)_{n=1}^\infty) = \bigcup_{n=1}^\infty U(a_1, \dots, a_n)$,*

(3) *$U(a_1, \dots, a_n) = U(a_1, \dots, a_{n-1}) \cup \{f(a_n) : f(x) \in T(a_1, \dots, a_{n-1}, x)\}$ for every $n \in \mathbb{N}$, and*

(4) *for every $n \in \mathbb{N}$ and $f(x) \in T(a_1, \dots, a_{n-1}, x)$, $f(1) \in U(a_1, \dots, a_{n-1})$.*