FOUNDATIONS of DIFFERENTIABLE MANIFOLDS and LIE GROUPS

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Editor's Preface

This textbook fills a long-standing gap. The beginning graduate student finds it hard to learn the basic material on differentiable manifolds. All the books he is referred to give a cursory treatment and quickly move on to more specialized topics. For this reason, Professor Warner's book is especially welcome. Here is a clear, detailed, and careful development of the fundamental facts on manifold theory and Lie groups. Numerous problems extend the theory and help the student master the subject. An added bonus is the sheaf-theoretic proof of the de Rham theorem and an elementary proof of the Hodge theorem. As far as I know, the latter proof is the only one in the literature easily accessible to the novice in analysis.

I. M. Singer

Preface

This book provides the necessary foundation for students interested in any of the diverse areas of mathematics which require the notion of a differentiable manifold. It is designed as a beginning graduate-level textbook and presumes a good undergraduate training in algebra and analysis plus some knowledge of point set topology, covering spaces, and the fundamental group. It is also intended for use as a reference book since it includes a number of items which are difficult to ferret out of the literature, in particular, the complete and self-contained proofs of the fundamental theorems of Hodge and de Rham.

The core material is contained in Chapters 1, 2, and 4. This includes differentiable manifolds, tangent vectors, submanifolds, implicit function theorems, vector fields, distributions and the Frobenius theorem, differential forms, integration, Stokes' theorem, and de Rham cohomology.

Chapter 3 treats the foundations of Lie group theory, including the relationship between Lie groups and their Lie algebras, the exponential map, the adjoint representation, and the closed subgroup theorem. Many examples are given, and many properties of the classical groups are derived. The chapter concludes with a discussion of homogeneous manifolds. The standard reference for Lie group theory for over two decades has been Chevalley's *Theory of Lie Groups*, to which I am greatly indebted.

For the de Rham theorem, which is the main goal of Chapter 5, axiomatic sheaf cohomology theory is developed. In addition to a proof of the strong form of the de Rham theorem—the de Rham homomorphism given by integration is a ring isomorphism from the de Rham cohomology ring to the differentiable singular cohomology ring—it is proved that there are canonical isomorphisms of all the classical cohomology theories on manifolds. The pertinent parts of all these theories are developed in the text. The approach which I have followed for axiomatic sheaf cohomology is due to H. Cartan, who gave an exposition in his *Séminaire* 1950/1951.

For the Hodge theorem, a complete treatment of the local theory of elliptic operators is presented in Chapter 6, using Fourier series as the basic tool. Only a slight acquaintance with Hilbert spaces is presumed. I wish to thank Jerry Kazdan, who spent a large portion of the summer of 1969 educating me to the whys and wherefores of inequalities and who provided considerable assistance with the preparation of this chapter. I also benefited from notes on lectures by J. J. Kohn and Stephen Andrea, from several papers of Louis Nirenberg, and from *Partial Differential*

Equations by Bers, John, and Schechter, which the reader might wish to consult for further references to the literature.

At the end of each chapter is a set of exercises. These are an integral part of the text. Often where a claim in a chapter has been left to the reader, there is a reminder in the Exercises that the reader should provide a proof of the claim. Some exercises are routine and test general understanding of the chapter. Many present significant extensions of the text. In some cases the exercises contain major theorems. Two notable examples are properties of the eigenfunctions of the Laplacian and the Peter-Weyl theorem, which are developed in the Exercises for Chapter 6. Hints are provided for many of the difficult exercises.

There are a few notable omissions in the text. I have not treated complex manifolds, although the sheaf theory developed in Chapter 5 will provide the reader with one of the basic tools for the study of complex manifolds. Neither have I treated infinite dimensional manifolds, for which I refer the reader to Lang's *Introduction to Differentiable Manifolds*, nor Sard's theorem and imbedding theorems, which the reader can find in Sternberg's *Lectures on Differential Geometry*.

Several possible courses can be based on this text. Typical one-semester courses would cover the core material of Chapters 1, 2, and 4, and then either Chapter 3 or 5 or 6, depending on the interests of the class. The entire text can be covered in a one-year course.

Students who wish to continue with further study in differential geometry should consult such advanced texts as *Differential Geometry and Symmetric Spaces* by Helgason, *Geometry of Manifolds* by Bishop and Crittenden, and *Foundations of Differential Geometry* (2 vols.) by Kobayashi and Nomizu.

I am happy to express my gratitude to Professor I. M. Singer, from whom I learned much of the material in this book and whose courses have always generated a great excitement and enthusiasm for the subject.

Many people generously devoted considerable time and effort to reading early versions of the manuscript and making many corrections and helpful suggestions. I particularly wish to thank Manfredo do Carmo, Jerry Kazdan, Stuart Newberger, Marc Rieffel, John Thorpe, Nolan Wallach, Hung-Hsi Wu, and the students in my classes at the University of California at Berkeley and at the University of Pennsylvania. My special thanks to Jeanne Robinson, Marian Griffiths, and Mary Ann Hipple for their excellent job of typing, and to Nat Weintraub of Scott, Foresman and Company for his cooperation and excellent guidance and assistance in the final preparation of the manuscript.

Frank Warner

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1

MANIFOLDS

After establishing some notational conventions which will be used throughout the book, we will begin with the notion of a differentiable manifold. These are spaces which are locally like Euclidean space and which have enough structure so that the basic concepts of calculus can be carried over. In this first chapter we shall primarily be concerned with the analogs and implications for manifolds of the fundamental theorems of differential calculus. Later, in Chapter 4, we shall consider the theory of integration on manifolds.

From the notion of directional derivative in Euclidean space we will obtain the notion of a tangent vector to a differentiable manifold. We will study mappings between manifolds and the effect that mappings have on tangent vectors. We will investigate the implications for mappings of manifolds of the classical inverse and implicit function theorems. We will see that the fundamental existence and uniqueness theorems for ordinary differential equations translate into existence and uniqueness statements for integral curves of vector fields. The chapter closes with the Frobenius theorem, which pertains to the existence and uniqueness of integral manifolds of involutive distributions on manifolds.

PRELIMINARIES

1.1 Some Basic Notation and Terminology Throughout this text we will describe sets either by listings of their elements, for example

$$\{a_1, \ldots, a_n\},$$

or by expressions of the form

$$\{x: P\},$$

which denote the set of all x satisfying property P. The expression $a \in A$ means that a is an *element* of the set A. If a set A is a *subset* of a set B (that is, $a \in B$ whenever $a \in A$), we write $A \subset B$. If $A \subset B$ and $B \subset A$, then A *equals* B, denoted $A = B$. The negations of $\in$, $\subset$ and $=$ are denoted by $\notin$, $\not\subset$, and $\neq$ respectively. A set A is a *proper subset* of B if $A \subset B$ but $A \neq B$.

We will denote the *empty set* by $\varnothing$. We will often denote a collection $\{U_\alpha : \alpha \in A\}$ of sets U_α indexed by the set A simply by $\{U_\alpha\}$ if explicit mention of the index set is not necessary. The *union* of the sets in the collection $\{U_\alpha : \alpha \in A\}$ will be denoted $\bigcup_{\alpha \in A} U_\alpha$ or simply $\bigcup U_\alpha$. Similarly, their *intersection* will be denoted $\bigcap_{\alpha \in A} U_\alpha$ or simply $\bigcap U_\alpha$.

$$\bigcup_{\alpha \in A} U_\alpha = \{a : a \text{ belongs to some } U_\alpha\}.$$

$$\bigcap_{\alpha \in A} U_\alpha = \{a : a \text{ belongs to every } U_\alpha\}.$$

The expression $f: A \to B$ means that f is a *mapping* of the set A into the set B. When describing a mapping by describing its effect on individual elements, we use the special arrow $\mapsto$; thus "the mapping $m \mapsto f(m)$ of A into B" means that f is a mapping of the set A into the set B taking the element m of A into the element $f(m)$ of B. If $U \subset A$, then $f \mid U$ denotes the *restriction of f to U*, and $f(U) = \{b \in B : f(a) = b \text{ for some } a \in U\}$. If $C \subset B$, then $f^{-1}(C) = \{a \in A : f(a) \in C\}$. A mapping f is *one-to-one* (also denoted 1 : 1), or *injective*, if whenever a and b are distinct elements of A, then $f(a) \neq f(b)$. A mapping f is *onto*, or *surjective*, if $f(A) = B$.

If $f: A \to B$ and $g: C \to D$, then the *composition* $g \circ f$ is the map

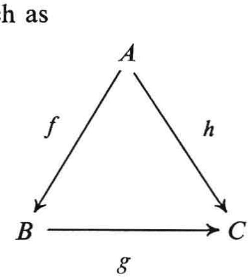
$$g \circ f : f^{-1}(B \cap C) \to D$$

defined by $g \circ f(a) = g(f(a))$ for every $a \in f^{-1}(B \cap C)$. For notational convenience, we shall not exclude the case in which $f^{-1}(B \cap C) = \varnothing$. That is, given any two mappings f and g, we shall consider their composition $g \circ f$ as being defined, with the understanding that the domain of $g \circ f$ may well be the empty set.

The *cartesian product $A \times B$ of two sets A and B* is the set of all pairs (a, b) of points $a \in A$ and $b \in B$. If $f: A \to C$ and $g: B \to D$, then the *cartesian product $f \times g$ of the maps f and g* is the map $(a, b) \mapsto (f(a), g(b))$ of $A \times B$ into $C \times D$.

We shall denote the *identity map* on any set by "id."

A diagram of maps such as



is called *commutative* if $g \circ f = h$.

We shall always use the term *function* to mean a mapping into the real numbers.

Let $d \geq 1$ be an integer, and let

$$\mathbb{R}^d = \{a\colon a = (a_1, \ldots, a_d) \text{ where the } a_i \text{ are real numbers}\}.$$

Then $\mathbb{R}^d$ is the *d-dimensional Euclidean space*. In the case $d = 1$, we denote the *real line* $\mathbb{R}^1$ simply by $\mathbb{R}$. The *origin* $(0, \ldots, 0)$ in Euclidean space of any dimension will be denoted 0. The notations $[a,b]$ and (a,b) denote as usual the intervals of the real line $a \leq t \leq b$ and $a < t < b$ respectively. The function $r_i\colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$r_i(a) = a_i,$$

where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, is called the ith (*canonical*) *coordinate function* on $\mathbb{R}^d$. The canonical coordinate function r_1 on $\mathbb{R}$ will be denoted simply by r. Thus $r(a) = a$ for each $a \in \mathbb{R}$. If $f\colon X \to \mathbb{R}^d$, then we let

$$f_i = r_i \circ f,$$

where f_i is called the ith *component function of* f.

If $f\colon \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, then we denote the *derivative of f at t* by

$$\left.\frac{d}{dr}\right|_t (f) = \left.\frac{df}{dr}\right|_t = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

If $f\colon \mathbb{R}^n \to \mathbb{R}$, if $1 \leq i \leq n$, and if $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, then we denote the *partial derivative of f with respect to r_i at t* by

$$\left.\frac{\partial}{\partial r_i}\right|_t (f) = \left.\frac{\partial f}{\partial r_i}\right|_t = \lim_{h \to 0} \frac{f(t_1, \ldots, t_{i-1}, t_i + h, t_{i+1}, \ldots, t_n) - f(t)}{h}.$$

If $p \in \mathbb{R}^d$, then $B_p(r)$ will denote the *open ball* of radius r about p. The open ball of radius r about the origin will be denoted simply by $B(r)$. $C(r)$ will denote the *open cube* with sides of length $2r$ about the origin in $\mathbb{R}^d$. That is,

$$C(r) = \{(a_1, \ldots, a_d) \in \mathbb{R}^d\colon |a_i| < r \text{ for all } i\}.$$

We shall use $\mathbb{C}$ to denote the *complex number field* and $\mathbb{C}^n$ to denote *complex n-space*,

$$\mathbb{C}^n = \{(z_1, \ldots, z_n)\colon z_i \in \mathbb{C} \text{ for } 1 \leq i \leq n\}.$$

Unless we indicate otherwise, we shall always use the term *neighborhood* in the sense of *open* neighborhood. If A is a subset of a topological space, its closure will be denoted by $\bar{A}$. If φ is a function on a topological space X, the *support* of φ is the subset of X defined by

$$\operatorname{supp} \varphi = \overline{\varphi^{-1}(\mathbb{R} - \{0\})}.$$

We use the *Kronecker index*

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a d-tuple of non-negative integers, then we set

$$[\alpha] = \sum \alpha_i,$$

$$\alpha! = \alpha_1!\,\alpha_2! \cdots \alpha_d!,$$

and

$$\frac{\partial^\alpha}{\partial r^\alpha} = \frac{\partial^{[\alpha]}}{\partial r_1^{\alpha_1} \cdots \partial r_d^{\alpha_d}}.$$

If $\alpha = (0, \ldots, 0)$, then we let

$$\frac{\partial^\alpha}{\partial r^\alpha}(f) = f.$$

DIFFERENTIABLE MANIFOLDS

1.2 Definitions Let $U \subset \mathbb{R}^d$ be open, and let $f\colon U \to \mathbb{R}$. We say that f is *differentiable of class* C^k *on* U (or simply that f is C^k), for k a non-negative integer, if the partial derivatives $\partial^\alpha f/\partial r^\alpha$ exist and are continuous on U for $[\alpha] \leq k$. In particular, f is C^0 iff f is continuous. If $f\colon U \to \mathbb{R}^n$, then f is *differentiable of class* C^k if each of the component functions $f_i = r_i \circ f$ is C^k. We say that f is C^∞ if it is C^k for all $k \geq 0$.

1.3 Definitions A *locally Euclidean space* M *of dimension* d is a Hausdorff topological space M for which each point has a neighborhood homeomorphic to an open subset of Euclidean space $\mathbb{R}^d$. If φ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^d$, φ is called a *coordinate map*, the functions $x_i = r_i \circ \varphi$ are called the *coordinate functions*, and the pair (U, φ) (sometimes denoted by $(U, x_1, \ldots, x_d)$) is called a *coordinate system.* A coordinate system (U, φ) is called a *cubic coordinate system* if $\varphi(U)$ is an open cube about the origin in $\mathbb{R}^d$. If $m \in U$ and $\varphi(m) = 0$, then the coordinate system is said to be *centered at m.*

1.4 Definitions A *differentiable structure* $\mathscr{F}$ *of class* C^k $(1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha)\colon \alpha \in A\}$ satisfying the following three properties:

(a) $\bigcup_{\alpha \in A} U_\alpha = M.$

(b) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^k for all $\alpha, \beta \in A$.

(c) The collection $\mathscr{F}$ is maximal with respect to (b); that is, if (U, φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \varphi) \in \mathscr{F}$.

If $\mathscr{F}_0 = \{(U_\alpha, \varphi_\alpha): \alpha \in A\}$ is any collection of coordinate systems satisfying properties (a) and (b), then there is a unique differentiable structure $\mathscr{F}$ containing $\mathscr{F}_0$. Namely, let

$$\mathscr{F} = \{(U,\varphi): \varphi \circ \varphi_\alpha^{-1} \text{ and } \varphi_\alpha \circ \varphi^{-1} \text{ are } C^k \text{ for all } \varphi_\alpha \in \mathscr{F}_0\}.$$

Then $\mathscr{F}$ contains $\mathscr{F}_0$, clearly satisfies (a), and it is easily checked that $\mathscr{F}$ satisfies (b). Now $\mathscr{F}$ is maximal by construction, and so $\mathscr{F}$ is a differentiable structure containing $\mathscr{F}_0$. Clearly $\mathscr{F}$ is the unique such structure.

We mention two other fundamental types of differentiable structures on locally Euclidean spaces, types that we shall not treat in this text, namely, the structure of class C^ω and the complex analytic structure. For a *differentiable structure of class* C^ω, one requires that the compositions in (b) are locally given by convergent power series. For a *complex analytic structure* on a $2d$-dimensional locally Euclidean space, one requires that the coordinate systems have range in complex d-space $\mathbb{C}^d$ and overlap holomorphically.

A *d-dimensional differentiable manifold of class* C^k (similarly C^ω or complex analytic) is a pair $(M,\mathscr{F})$ consisting of a d-dimensional, second countable, locally Euclidean space M together with a differentiable structure $\mathscr{F}$ of class C^k. We shall usually denote the differentiable manifold $(M,\mathscr{F})$ simply by M, with the understanding that when we speak of the "differentiable manifold M" we are considering the locally Euclidean space M with some given differentiable structure $\mathscr{F}$. Our attention will be restricted solely to the case of class C^∞, so by *differentiable* we will always mean *differentiable of class* C^∞. We also use the terminology *smooth* to indicate differentiability of class C^∞. We often refer to differentiable manifolds simply as *manifolds*, with differentiability of class C^∞ always implicitly assumed. A manifold can be viewed as a triple consisting of an underlying point set, a second countable locally Euclidean topology for this set, and a differentiable structure. If X is a set, by a *manifold structure on* X we shall mean a choice of both a second countable locally Euclidean topology for X and a differentiable structure.

Even though we shall restrict our attention to the C^∞ case, many of our theorems do, however, have C^k versions for $k < \infty$, which are essentially no more complicated than the ones we shall obtain. They simply require that one keep track of degrees of differentiability, for differentiating a C^k function may only yield a function of class C^{k-1} if $1 \leq k < \infty$.

Unless we indicate otherwise, we shall always use M and N to denote differentiable manifolds, and M^d will indicate that M is a manifold of dimension d.

1.5 Examples

(a) The standard differentiable structure on Euclidean space $\mathbb{R}^d$ is obtained by taking $\mathscr{F}$ to be the maximal collection (with respect to 1.4(b)) containing $(\mathbb{R}^d,i)$, where $i: \mathbb{R}^d \to \mathbb{R}^d$ is the identity map.

(b) Let V be a finite dimensional real vector space. Then V has a natural manifold structure. Indeed, if $\{e_i\}$ is a basis of V, then the elements of the dual basis $\{r_i\}$ are the coordinate functions of a global coordinate system on V. Such a global coordinate system uniquely determines a differentiable structure $\mathscr{F}$ on V. This differentiable structure is independent of the choice of basis, since different bases give C^∞ overlapping coordinate systems. In fact, the change of coordinates is given simply by a constant non-singular matrix.

(c) Complex n-space $\mathbb{C}^n$ is a real $2n$-dimensional vector space, and so, by Example (b), has a natural structure as a $2n$-dimensional real manifold. If $\{e_i\}$ is the canonical complex basis in which e_i is the n-tuple consisting of zeros except for a 1 in the ith spot, then

$$\{e_1, \ldots, e_n, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_n\}$$

is a real basis for $\mathbb{C}^n$, and its dual basis is the canonical global coordinate system on $\mathbb{C}^n$.

(d) The *d-sphere* is the set

$$S^d = \{a \in \mathbb{R}^{d+1}: \sum_{i=1}^{d+1} a_i^2 = 1\}.$$

Let $n = (0, \ldots, 0, 1)$ and $s = (0, \ldots, 0, -1)$. Then the standard differentiable structure on S^d is obtained by taking $\mathscr{F}$ to be the maximal collection containing $(S^d - n, p_n)$ and $(S^d - s, p_s)$, where p_n and p_s are stereographic projections from n and s respectively.

(e) An open subset U of a differentiable manifold $(M, \mathscr{F}_M)$ is itself a differentiable manifold with differentiable structure

$$\mathscr{F}_U = \{(U_\alpha \cap U, \varphi_\alpha \mid U_\alpha \cap U): (U_\alpha, \varphi_\alpha) \in \mathscr{F}_M\}.$$

Unless specified otherwise, open subsets of differentiable manifolds will always be given this natural differentiable structure.

(f) The *general linear group* $Gl(n, \mathbb{R})$ is the set of all $n \times n$ non-singular real matrices. If we identify in the obvious way the points of $\mathbb{R}^{n^2}$ with $n \times n$ real matrices, then the determinant becomes a continuous function on $\mathbb{R}^{n^2}$. $Gl(n, \mathbb{R})$ receives a manifold structure as the open subset of $\mathbb{R}^{n^2}$ where the determinant function does not vanish.

(g) *Product manifolds.* Let $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ be differentiable manifolds of dimensions d_1 and d_2 respectively. Then $M_1 \times M_2$ becomes a differentiable manifold of dimension $d_1 + d_2$, with differentiable structure $\mathscr{F}$ the maximal collection containing

$$\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta: U_\alpha \times V_\beta \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}): (U_\alpha, \varphi_\alpha) \in \mathscr{F}_1, (V_\beta, \psi_\beta) \in \mathscr{F}_2\}.$$

1.6 Definitions Let $U \subset M$ be open. We say that $f: U \to \mathbb{R}$ is a *C^∞ function on U* (denoted $f \in C^\infty(U)$) if $f \circ \varphi^{-1}$ is C^∞ for each coordinate map φ on M. A continuous map $\psi: M \to N$ is said to be *differentiable of class C^∞* (denoted $\psi \in C^\infty(M,N)$ or simply $\psi \in C^\infty$) if $g \circ \psi$ is a C^∞ function on ψ^{-1}(domain of g) for all C^∞ functions g defined on open sets in N. Equivalently, the continuous map ψ is C^∞ if and only if $\varphi \circ \psi \circ \tau^{-1}$ is C^∞ for each coordinate map τ on M and φ on N.

Clearly the composition of two differentiable maps is again differentiable. Observe that a mapping $\psi: M \to N$ is C^∞ if and only if for each $m \in M$ there exists an open neighborhood U of m such that $\psi \mid U$ is C^∞.

THE SECOND AXIOM OF COUNTABILITY

The second axiom of countability has many consequences for manifolds. Among them, manifolds are normal, metrizable, and paracompact. Paracompactness implies the existence of partitions of unity, an extremely useful tool for piecing together global functions and structures out of local ones, and conversely for representing global structures as locally finite sums of local ones. After giving the necessary definitions, we shall give a simple direct proof of paracompactness for manifolds, and shall then derive the existence of partitions of unity. It is evident that manifolds are regular topological spaces and their normality follows easily from this and the paracompactness. We shall leave the proof that manifolds are normal as an exercise. For the fact that manifolds are metrizable, see [13].

1.7 Definitions A collection $\{U_\alpha\}$ of subsets of M is a *cover* of a set $W \subset M$ if $W \subset \bigcup U_\alpha$. It is an *open cover* if each U_α is open. A subcollection of the U_α which still covers is called a *subcover*. A *refinement* $\{V_\beta\}$ of the cover $\{U_\alpha\}$ is a cover such that for each β there is an α such that $V_\beta \subset U_\alpha$. A collection $\{A_\alpha\}$ of subsets of M is *locally finite* if whenever $m \in M$ there exists a neighborhood W_m of m such that $W_m \cap A_\alpha \neq \varnothing$ for only finitely many α. A topological space is *paracompact* if every open cover has an open locally finite refinement.

1.8 Definition A *partition of unity on M* is a collection $\{\varphi_i: i \in I\}$ of C^∞ functions on M such that

(a) The collection of supports $\{\text{supp } \varphi_i: i \in I\}$ is locally finite.

(b) $\sum_{i \in I} \varphi_i(p) = 1$ for all $p \in M$, and $\varphi_i(p) \geq 0$ for all $p \in M$ and $i \in I$.

A partition of unity $\{\varphi_i: i \in I\}$ is *subordinate* to the cover $\{U_\alpha: \alpha \in A\}$ if for each i there exists an α such that $\text{supp } \varphi_i \subset U_\alpha$. We say that it is subordinate to the cover $\{U_i: i \in I\}$ *with the same index set as the partition of unity* if $\text{supp } \varphi_i \subset U_i$ for each $i \in I$.