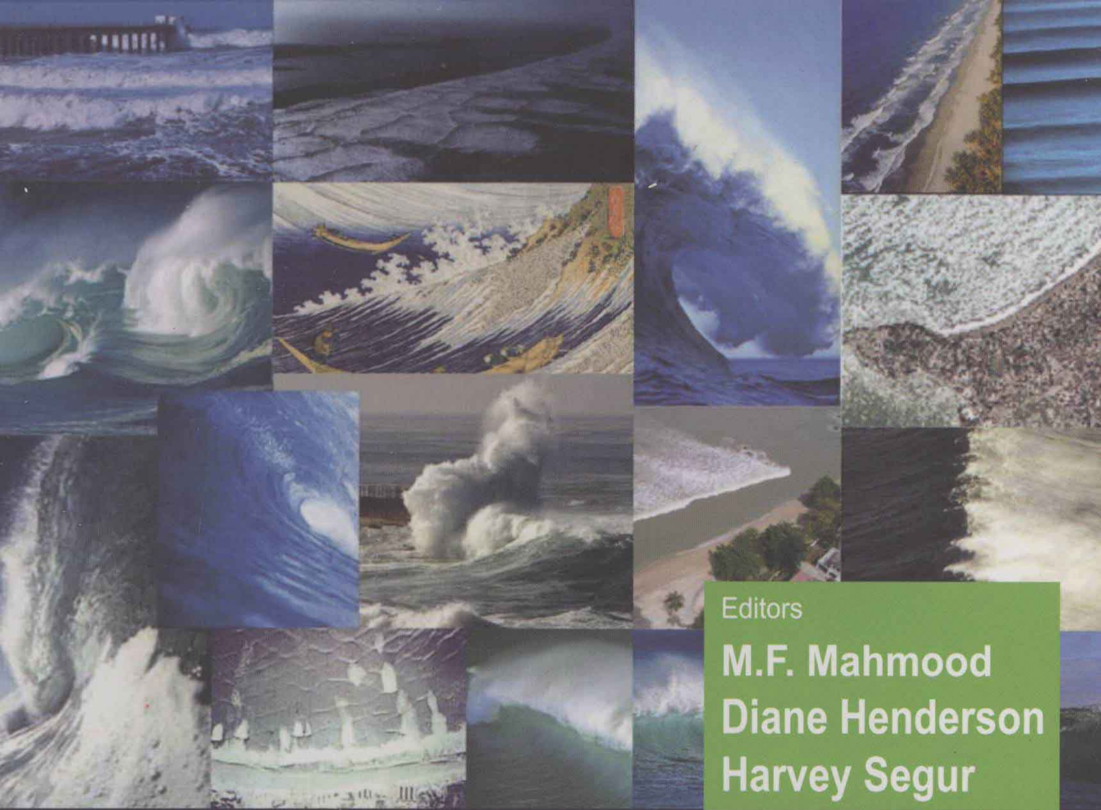

Editors

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Proceedings of the Conference on

WATER WAVES

THEORY AND EXPERIMENT

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PREFACE

This volume contains papers presented at the NSF/CBMS Conference on Water Waves: Theory and Experiment. The five-day Regional Conference was held on the main campus of Howard University from the 13^{th} to the 18^{th} of May 2008. Professor Harvey Segur was the principal lecturer of this conference. He delivered lectures on aspects of the mathematical theory of water waves. Professor Diane Henderson has designed experiments to support and complement the lectures. Professor M. F. Mahmood was the Principal Investigator (PI) and organizer of this conference.

The theory of water waves is appealing as a concrete prototype of a nontrivial dynamical system — the system of partial differential equations is nonlinear and its phase space is infinite-dimensional. If we neglect dissipation, then it is also Hamiltonian. It exhibits naturally several concepts that have been developed in nonlinear dynamics and nonlinear wave propagation: linear and nonlinear instabilities, deterministic chaos, resonant triad and quartet interactions, solitons and complete integrability. Both deterministic and statistical approximations of water waves have been used fruitfully. Best of all, we can observe these concepts in physical experiments that can be carried out without much special equipment.

We believe that the collected papers, by presenting important recent developments, should offer a valuable source of inspiration for new entrants in the area, including post-doctoral fellows and graduate students in pure and applied mathematics, and established researchers in the field.

ACKNOWLEDGMENT

We would like to express our deepest gratitude to the National Science Foundation. This conference would not have been possible without its support.

We also wish to thank all the participants, authors, referees, colleagues for their various and irreplaceable contributions, including Sean Brooks, Knox, Fahad, Zafar and Asad for their constant help during the preparation of this volume. As well, we would like to express our gratitude to Jawad Al-Khal and Miriam Ahmed for their art and design of the poster for the conference and the book cover. Finally, we acknowledge Howard University, in particular the Office of Provost, for its encouragement and generous support towards the publication of this proceedings.

M. F. Mahmood, D. Henderson & H. Segur
November 6, 2009

CONTENTS

GRAVITY INDUCED DISPERSION FOR NEARLY FLAT VORTEX SHEETS

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Using techniques from the theory of oscillatory integrals, we prove rigorous estimates which show that the linearization of the vortex sheet equations of motion about a quiescent state disperse under certain circumstances. Such dispersion is only possible only through the joint effects of surface tension (which damps high frequency modes) and gravitation (which damps low frequency modes).

Keywords: vortex sheets; oscillatory integrals; water waves; dispersive estimates.

1. Dispersive effects in vortex sheets

Consider the flow of a pair of two-dimensional ideal fluids which shear past one another along an interface on which surface tension acts — that is, we have a "vortex sheet" system. Suppose that gravity acts on this system and that the lighter fluid is above the heavier. Also, suppose that the velocity field is curl-free at all points in the fluids not on the interface. The fluids cover all of $\mathbf{R}^2$. The equations of motion for this system are well-known (*e.g.* [5]):

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = 0 \quad \text{in the fluid domain}$$

$$\mathbf{u} = \nabla\phi, \quad \Delta\phi = 0 \quad \text{in the fluid domain}$$

$$[\mathbf{u} \cdot \mathbf{n}] = 0 \quad \text{on the surface}$$

$$[p] = \tau K \quad \text{on the surface.}$$

Here, $\mathbf{u}$ is the velocity field, p is pressure, $\mathbf{k} = (0,1)^t$, ρ is density, g is gravitational acceleration, ϕ is the velocity potential, $\mathbf{n}$ is the upward unit normal to the interface, K is curvature of the interface, τ is the constant of surface tension and $[Q]$ represents the jump of a quantity "Q" across the interface. The scenario which occurs when upper fluid is replaced with a vacuum is called the "water wave" problem.

The vortex sheet and water wave problems are typically referred to as "dispersive systems." One characteristic typical of dispersive systems is that the amplitude of solutions decays at an algebraic rate, even though the full system may conserves certain norms (*e.g.* [3]). The main purpose of our work [13] is pin down in full technical detail when this sort of decay should be expected for linearized vortex sheets and at what rate this decay takes place. Estimates of this sort can be extremely useful for proving the global-in-time existence[a] of solutions of the nonlinear problem (*e.g.* [2]), as well as being of interest in and of themselves.

Since the velocity potential solves Laplace's equation, one can reformulate this problem entirely in terms of functions defined on the interface. There are several ways to do this, and we choose to use a method developed in [7]. The interface can be represented as a curve in $\mathbf{R}^2$ parameterized by arclength. Call this parameter α, and suppose that

$$\theta(\alpha,t) = \text{ tangent angle w.r.t. horizontal of the interface at } (\alpha,t)$$

and

$$\gamma(\alpha,t) = \text{ jump in tangential velocity at } (\alpha,t).$$

We omit the exact equations of motion in terms of θ and γ; they can be found in [1]. This system is in equilibrium when the two fluids shear past one another along a perfectly flat interface. That is, when $\theta(\alpha,t) = 0$ and $\gamma(\alpha,t) = \bar{\gamma}$. Linearizing about this state results in the system:

$$\begin{aligned} \partial_t\theta &= \frac{1}{2}\mathcal{H}(\partial_\alpha\gamma) \\ \partial_t\gamma &= 2\tau\partial_\alpha^2\theta + \frac{\bar{\gamma}^2}{2}\mathcal{H}(\partial_\alpha\theta) - \frac{A}{2}\bar{\gamma}\partial_\alpha\gamma - 2Ag\theta, \end{aligned} \tag{1}$$

where $\mathcal{H}$ (the Hilbert tranform) is given by the singular integral

$$\mathcal{H}f(\alpha) := \frac{1}{\pi}P.V.\int_{\mathbf{R}} \frac{f(\alpha')}{\alpha-\alpha'}d\alpha'.$$

The constant $A = (\rho_{lower} - \rho_{upper})/(\rho_{lower} + \rho_{upper})$ is positive.

---

[a] Local existence of solutions is known for arbitrary initial data, see [1, 8, 14], for instance.

Looking for plane wave solutions $e^{i(\xi\alpha-\omega(\xi)t)}\mathbf{v}$ to (1) yields the dispersion relation (see [10])

$$\omega(\xi) = -\frac{A\bar{\gamma}}{4} \pm \lambda(\xi)$$

with

$$\lambda^2(\xi) := \tau\,|\xi|^3 - \frac{\bar{\gamma}^2}{4}\left(1-\frac{A^2}{4}\right)|\xi|^2 + Ag\,|\xi|. \tag{2}$$

Note that if $\lambda^2(\xi) < 0$ at wave number ξ, then the dispersion relation is imaginary and we expect exponential growth of that mode. Notice that the first term of $\lambda^2(\xi)$ represents the contribution from surface tension, the second that of shearing and the last comes from gravity. Importantly the signs on surface tension and gravity terms are positive, while the shear term is negative. Therefore we see that in the absence of surface tenstion, $\lambda^2(\xi)$ become negative for large wave numbers, and that in the absence of gravitation the same is true for low wave numbers. That is to say **dispersive decay can only occur if gravitation and surface tension are sufficiently strong relative to the ambient shear.**

More quantitatively, we have $\lambda^2(\xi) > 0$ for all ξ when

$$\frac{\bar{\gamma}^4}{8}\left(1-\frac{A^2}{4}\right)^2 < 4Ag\tau. \tag{3}$$

This condition guarantees there will be no exponential growth of solutions, but it does not immediately demonstrate that decay occurs. To see this, we first compute the solution of (1) by means of the Fourier transform. It is:

$$\begin{aligned}
\widehat{\theta}(\xi,t) = e^{ic_1\xi t}\Bigg[&\left(-\frac{ic_1\xi\sin(\lambda(\xi)t)}{\lambda(\xi)} + \cos(\lambda(\xi)t)\right)\widehat{\theta}_0(\xi) \\
&+ \frac{|\xi|\sin(\lambda(\xi)t)}{\lambda(\xi)}\widehat{\gamma}_0(\xi)\Bigg] \\
\widehat{\gamma}(\xi,t) = e^{ic_1\xi t}\Bigg[&\frac{c_1^2-\lambda^2(\xi)}{|\xi|\lambda(\xi)}\sin(\lambda(\xi)t)\widehat{\theta}_0(\xi) \\
&+ \left(-\frac{ic_1\xi\sin(\lambda(\xi)t)}{\lambda(\xi)} + \cos(\lambda(\xi)t)\right)\widehat{\gamma}_0(\xi)\Bigg]
\end{aligned} \tag{4}$$

where $c_1 = -A\bar{\gamma}/4$.

Thus estimating the amplitude of the solution boils down to estimating operators of the form:

$$S(t)f := \mathfrak{F}[e^{i\lambda(\cdot)t}\sigma(\cdot)\widehat{f}(\cdot)] = \int_{\mathbf{R}} e^{i(\xi\alpha+\lambda(\xi)t)}\sigma(\xi)\widehat{f}(\xi)d\xi.$$

(Here, the function $\sigma(\xi)$ represents the sundry multipliers that appear in (4).) Such estimates fall within the purview of harmonic analysis; the method of stationary phase is the typical way of controlling L^∞ norms of such oscillatory integrals. We have two concerns when estimating $S(t)f$. The first is, at what rate does it decay (if at all)? The second is, what is the least restrictive space in which we can place f?

Roughly speaking, the method of stationary phase says the following (see [11]): Suppose on an interval $[a,b]$ (possibly infinite) we have a point ξ_{stat} (a stationary point) at which

$$h^{(j)}(\xi_{stat}) = 0$$

for $j = 1, ..., n-1$ but that

$$h^{(n)}(\xi_{stat}) \neq 0.$$

Then

$$\left| \int_a^b e^{ih(\xi)t} d\xi \right| \leq Ct^{-1/n}.$$

The constant C is proportional to $\left(\min_{[a,b]} \left|h^{(n)}\right|\right)^{1/n}$.

For our problem we need to control, for instance,

$$\left| \int_0^\infty e^{i(\xi\alpha+\lambda(\xi)t)} d\xi \right| = \left| \int_0^\infty e^{it(\kappa\alpha+\lambda(\xi)t)} d\xi \right|$$

uniformly in $\kappa = x/t$. Supposing that the dispersive decay condition (3) is satisfied, observe the following facts about $\lambda(\xi)$:

i) $\lambda'(\xi) \sim \xi^{-1/2}$ for $\xi \sim 0$.

ii) $\lambda'(\xi) \sim \xi^{1/2}$ as $\xi \to \infty$.

Since $\lambda'(\xi)$ diverges near the origin and at infinity, there is a minimum of $\lambda'(\xi)$ some point ξ_{stat}. So if we have $\kappa = \kappa_{stat} := -\lambda'(\xi_{stat})$ then we have:

$$\kappa_{stat} + \lambda'(\xi_{stat}) = \lambda''(\xi_{stat}) = 0.$$

It happens that $\lambda'''(\xi_{stat}) \neq 0$. And so the stationary phase argument indicates that this integral should decay like $Ct^{-1/3}$. (Note that $|\kappa_{stat}|$ corresponds to the rate of the slowest "ripple" one sees when one throws a pebble into a pond, see [12].) This is in fact the case, but there is the complication that $\lambda''(\xi) \sim |\xi|^{-1/2}$ for $\xi \to \infty$. This means that the constant one gets from the stationary phase argument is infinite if we really work with the integral over all of $\mathbf{R}^+$.

We can bypass this problem by truncating our integrals in Fourier space, (see [6]), and this leads us into the second issue: what space is f in? We have (if $\sigma = 1$)

$$|S(t)f| \leq \left| \int_{|\xi| \leq t^\beta} e^{i(\xi\alpha + \lambda(\xi)t)} \widehat{f}(\xi) d\xi \right| + \left| \int_{|\xi| \geq t^\beta} e^{i(\xi\alpha + \lambda(\xi)t)} \widehat{f}(\xi) d\xi \right| .$$

Here $\beta > 0$ is a carefully chosen constant. We control the first integral using the stationary phase argument outlined above. The second term can be controlled if we assume some regularity of f. That is, if we know $\widehat{f}(\xi)$ decays for large ξ. If this decay is fast enough, since the second integral is being taken over smaller and smaller sets as t increases, this integral decays as well. Pursuing this course of action shows that (for $\sigma = 1$).

$$\|S(t)f\|_{L^\infty} \leq Ct^{-1/3} \|f\|_{H^1 \cap L^1} .$$

The sundry multipliers σ that appear in (4) may grow as ξ goes to infinity, and so their inclusion will correspondingly change the regularity required. They do not change the fact that the rate of decay is $t^{-1/3}$. We are able to substantially reduce the regularity requirement on f by working in Besov spaces, though the technical details are somewhat cumbersome for inclusion here.

We conclude this note with the following remarks:

i) If one considers the water wave problem with no surface tension, the decay rate increases from $t^{-1/3}$ to $t^{-1/2}$, though the regularity requirements are more restrictive.

ii) Dispersive estimates like the ones described here are the foundation for proving both smoothing estimates (see [4]) and (when combined with *a priori* energy estimates) Strichartz-type estimates (see [9]), both of importance for passing to the nonlinear problem.

iii) Global existence results for the nonlinear problem typically require a decay rate faster than $t^{-1/2}$. This seems to spell doom for our long term goal. However, if one considers three-dimensional fluids, the decay rate increases by $t^{-1/2}$. That is to say, to $t^{-5/6}$ with surface tension and to t^{-1} without.

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AN EXPERIMENTAL STUDY OF LOW-FREQUENCY WAVES GENERATED BY RANDOM GRAVITY WAVES IN SHOALING WATER

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A laboratory study of the nonlinearity of random waves (the primary waves) and low-frequency waves induced by the primary waves on inclined beaches is presented. The experiments were carried out in a wave flume that is 65 m long, 1.2 m wide with a water depth of 0.7 m. Three plane beaches with different slopes (1/20, 1/30 and 1/40) were used in separated experiments. Incident primary waves with a Person-Moskowitz (PM) spectrum were mechanically generated with significant wave height ranging from 0.047 to 0.125 m. The time series of surface elevations at different water depths along the beach were simultaneously recorded. Low-frequency waves are obtained from the measured wave data with a low-pass filter. The results show that the primary waves on the beach are highly nonlinear. The surface skewness and kurtosis, two statistical measures of wave nonlinearity, are functions of a nondimensional parameter H_s/d, where H_s is the local significant wave height and d is the water depth. The spectra of low-frequency waves in shoaling region are affected by beach slope and the energy of incident primary waves. The energy ratio between low-frequency waves and primary waves strongly correlates to the local surface skewness. The growth (dissipation) rate of low-frequency waves on beaches is controlled by the Iribarren number, $\xi = \beta/(H_0/L_0)^{1/2}$, where β is the beach slope, H_0 and L_0 are the significant wave height and wavelength of incident primary waves, respectively.

Keywords: Wave nonlinearity, low-frequency waves, shoaling.

1. Introduction

Under a normal circumstance, incident seas and swells in deep water are regarded as linear random processes. As these waves propagate into shallow water, the wave profiles show strong vertical asymmetry with sharp crests and round troughs. (As the wave approaches breaking, the profile also shows a strong horizontal asymmetry which is not discussed in this study). The probability distribution of sea surface elevations gradually deviates from the Gaussian function with decreasing water depth. It is well known that the non-Gaussian characteristics of sea surface elevations are related to the nonlinear dynamics of random waves. Surface skewness and kurtosis, corresponding to the 3rd and 4th order moments of the probability distribution of surface elevation, respectively, can be used to statistically measure the nonlinearity of waves (Phillips, 1961; Longuet-Higgins, 1963; Huang and Long, 1980; Srokosz and Longuet-Higgins, 1986; and others).

The surface skewness and kurtosis of random gravity waves in shallow water has been extensively studied in the laboratory and field. A laboratory experiment of mechanically generated random waves over an inclined beach with slope 1/20 was given by Mase (1989). The results of surface skewness (λ_3) versus d/H_0, where d is the water depth and H_0 is the significant wave height of incident random waves, show that as d/H_0 decreases, λ_3 monotonically increases and achieves a maximum at the outer breakpoint (the furthest offshore location of depth-induced wave breaking) and then decreases in breaking zone. Similar results of the surface skewness evolution on beaches were obtained in wind wave experiments (Ding et al, 1994b) and field measurements (Elgar and Guza; 1985, 1986) and compared with the predictions of nonlinear shoaling wave models proposed by Ding et al (1994a) and Freilich and Guza (1984), respectively.

Through the nonlinear wave-wave interaction, random gravity waves in shallow water may generate low-frequency waves. Munk (1949) and Tucker (1950) are among the first to observe low-frequency waves in the field near the shore. Further field investigations of low-frequency waves under different beach conditions have been given by a large number of researchers (Holman et al, 1978; Guza and Thornton, 1985; Elgar et al, 1992; Herbers et al, 1994; Masselink, 1995; Ruessink, 1998; Sheremet et al, 2002; Henderson and Bowen, 2002; Henderson et al, 2006 and others). General speaking, it has been found that the generation of low-frequency waves in shoaling region depends on the energy of incident gravity waves and the topography of beaches. For a sloping beach, the propagation of low-frequency waves in the shoreward direction is nonlinearly forced by incident short-wave groups with a phase lag of 180 degrees. After being reflected by the shoreline, low-frequency waves freely propagate in the seaward direction.

Longuet-Higgins and Stewart (1962, 1964) derived a theory to explain the field observations of Munk and Tucker. Following the work of Longuet-Higgins and Stewart, various models for second-order bound long waves in shallow water with uneven bottoms have been developed (Gallagher, 1971; Mei and Benmoussa, 1984; List, 1992; Madson et al, 1997; Janssen et al, 2003 and others). Comparisons with observations have shown diverse capabilities of these models in the prediction of low-frequency waves in shoaling region. However, large differences between theory and laboratory measurement still exist. For instance, the spectra predicted by the second-order long wave theory are only half of those measured in wave tank (see Fig.2 in Baldock and Huntly, 2002). Also, little attention has been given to the correlation between low-frequency waves and the statistical measure of gravity wave nonlinearity, for instance, the surface skewness mentioned above.

Low-frequency waves in shoaling water may be generated by a time-varying breakpoint mechanism proposed by Symonds et al (1982). In the theory of Symonds et al, long waves are directly radiated from the break-point position which oscillates with time. The frequency of the long waves approximately equals the frequency of short-wave groups. The theory is qualitatively supported by the laboratory experiments of Kostense (1984) and extended by Schaffer (1993) with short wave forcing within the surf zone.

A number of laboratory experiments of low-frequency waves on plane beaches have been recently given by two groups of people (Baldock et al, 2000; Baldock and Huntley, 2002; Baldock, 2006; Janssen et al, 2003; Battjes et al, 2004; van Dongeren et al, 2007). In these experiments, long waves were generated by the shoaling of incident gravity waves and/or the breaking of waves in group. Three types of incident gravity waves (bichromatic waves, random waves with uniform phase distribution and transient-focused waves) were used. Baldock and his coauthors (Baldock et al, 2000; Baldock and Huntley, 2002; Baldock, 2006) provide some evidences for the break-point generation mechanism. Battjes et al (2004) found that the shoaling regimes for the low-frequency waves can be characterized by a normalized beach slope parameter;

$$\beta_s = \frac{d_x}{\omega}\sqrt{\frac{g}{d}}, \tag{1}$$

where d_x is the beach slope, ω is the frequency of low-frequency waves, g is the acceleration due to gravity and d is the representative water depth. For large values of β_s (> 0.3), the incoming low-frequency waves are weakly enhanced, but strongly reflected from the shoreline . For small values of β_s