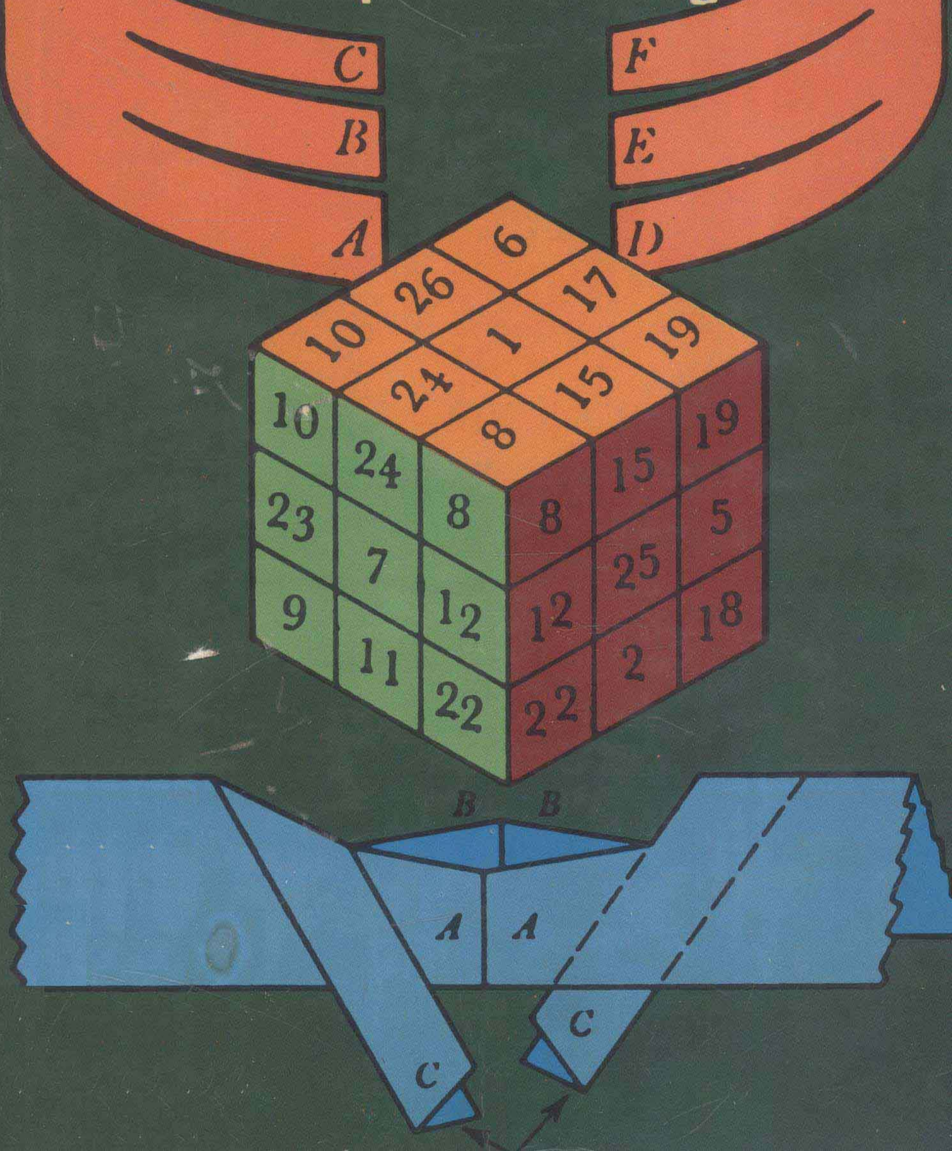
MATHEMATICAL DIVERSIONS

J.A.H. Hunter

Joseph S. Madachy



MATHEMATICAL DIVERSIONS

J. A. H. HUNTER

and

JOSEPH S. MADACHY

DOVER PUBLICATIONS, INC.
NEW YORK

Published in Canada by General Publishing Company, Ltd., 30 Lesmill Road, Don Mills, Toronto, Ontario.
Published in the United Kingdom by Constable and Company, Ltd., 10 Orange Street, London WC 2.

This Dover edition, first published in 1975, is an unabridged and revised republication of the work originally published by D. Van Nostrand Company, Inc., Princeton, New Jersey, in 1963, with a new preface by the authors.

International Standard Book Number: 0-486-23110-0
Library of Congress Catalog Card Number: 74-83619

Manufactured in the United States of America
Dover Publications, Inc.
180 Varick Street
New York, N. Y. 10014

DOVER BOOKS ON GAMES, RECREATIONS, MAGIC, ETC.

EASY-TO-DO ENTERTAINMENTS AND DIVERSIONS WITH CARDS, STRING, COINS, PAPER AND MATCHES, R. M. Abraham. (20921-0) $1.75

TRICKS AND AMUSEMENTS WITH CARDS, STRING, COINS, PAPER AND MATCHES, R. M. Abraham. (21127-4) $1.75

FUN WITH STRING FIGURES, W. W. Rouse Ball. (22809-6) $1.25

TRICKS, GAMES AND PUZZLES WITH MATCHES, Maxey Brooke. (20178-3) $1.00

COIN GAMES AND PUZZLES, Maxey Brooke. (22893-2) $1.25

HAND SHADOWS TO BE THROWN UPON THE WALL, Henry Bursill. (21779-5) $1.25

MORE HAND SHADOWS TO BE THROWN UPON THE WALL, Henry Bursill. (21384-6) $1.25

COWBOY ROPING AND ROPE TRICKS, Chester Byers. (21535-0) $1.50

THE SECRETS OF HOUDINI, J. C. Cannell. (22913-0) $2.50

PANORAMA OF MAGIC, Milbourne Christopher. (20774-9) $4.95

DALY'S BILLIARD BOOK, Maurice Daly. (22638-7) $2.95

JUDO TECHNIQUES AND TACTICS, Eric Dominy. (22310-8) $2.50

GAMES ANCIENT AND ORIENTAL AND HOW TO PLAY THEM, Edward Falkener. (20739-0) $3.50

MATHEMATICS, MAGIC AND MYSTERY, Martin Gardner. (20335-2) $2.00

THE COMPLETE KANO JIU-JITSU, Irving H. Hancock and Katsukuma Higashi. (20639-4) $4.00

TRICKS AND GAMES ON THE POOL TABLE, Fred Hermann. (21814-7) $1.25

WIN AT BACKGAMMON, Millard Hopper. (22894-0) $1.50

WIN AT CHECKERS, Millard Hopper. (20363-8) $1.75

HOUDINI ON MAGIC, Harry Houdini. (20384-0) $2.50

CARD MANIPULATIONS, Jean Hugard. (20539-8) $2.00

25 KITES THAT FLY, Leslie L. Hunt. (22550-X) $1.25

STRING FIGURES AND HOW TO MAKE THEM, Caroline F. Jayne. (20152-X) $3.00

MAGIC TRICKS AND CARD TRICKS, Wilfrid Jonson. (20909-1) $1.50

GO AND GO-MOKU, Edward Lasker. (20613-0) $2.50

(continued on back flap)

MATHEMATICAL DIVERSIONS

PREFACE TO THE DOVER EDITION

In this new Dover edition we have incorporated recent discoveries such as details of the new greatest known prime number. Apart from these, and the correction of certain errors, there have been no changes from the original text.

August 1974 THE AUTHORS

PREFACE TO THE FIRST EDITION

"Something old, and something new"! This seem to sum up what we have tried to achieve in this book. And with emphasis on the fun that the true lover of recreational mathematics finds in doing, rather than in reading about the doing!

Beyond that there is little need for introductory remarks so far as the content is concerned.

For much of the information on polyominoes we are indebted to S. W. Golomb's pioneering work. The valuable suggestions of a few kind people have been acknowledged individually in the text. For various other ideas, we would like to express our thanks in particular to the following:

Jean H. Anderson	Sinclair Grant
Stephen Barr	Guy Guillotte
A. G. Bradbury	R. B. McDunphy
Alan L. Brown	Derrick Murdoch
Spencer Earnshaw	John Waldorf

January, 1963 THE AUTHORS

A CATALOGUE OF SELECTED DOVER BOOKS IN ALL FIELDS OF INTEREST

A CATALOGUE OF SELECTED DOVER BOOKS IN ALL FIELDS OF INTEREST

(continued from front flap)

BRIDGE SQUEEZES COMPLETE; OR, WINNING END PLAY STRATEGY, Clyde E. Love. (21968-2) $2.75

THE 8TH BOOK OF TAN, Sam Loyd. (22011-7) $2.00

MAZES AND LABYRINTHS: THEIR HISTORY AND DEVELOPMENT, W. H. Matthews. (22614-X) $2.50

PUZZLE, QUIZ AND STUNT FUN, Jerome S. Meyer. (20337-9) $2.50

PAPER FOLDING FOR BEGINNERS, William D. Murray and Francis J. Rigney. (20713-7) $1.25

MAGIC AND SHOWMANSHIP: A HANDBOOK FOR CONJURERS, Henning Nelms. (22337-X) $3.00

FINGER PLAYS FOR NURSERY AND KINDERGARTEN, Emilie Poulsson. (22588-7) $1.50

TANGRAMS: 330 PUZZLES, Ronald C. Read. (21483-4) $1.50

BRIDGE CONVENTIONS, FINESSES AND COUPS, Terence Reese. (22631-X) $2.50

BIDDING A BRIDGE HAND, Terence Reese. (22830-4) $3.00

BRIDGE FOR BRIGHT BEGINNERS, Terence Reese. (22942-4) $1.75

PLAY BRIDGE WITH REESE, Térence Reese. (22313-2) $2.50

GEOMETRIC EXERCISES IN PAPER FOLDING, Sundara T. Row. (21594-6) $2.00

DUPLICATE BRIDGE, Alfred Sheinwold. (22741-3) $2.00

MAZES AND LABYRINTHS: A BOOK OF PUZZLES, Walter Shepherd. (20731-5) $1.35

BIG BOOK OF MAZES AND LABYRINTHS, Walter Shepherd. (22951-3) $2.00

INDIAN SIGN LANGUAGE, William Tomkins. (22029-X) $1.50

THE PRECISION BIDDING SYSTEM IN BRIDGE, C. C. Wei. (21171-1) $1.75

THE BOOK OF JACKS AND JACK GAMES: FOLLOW MY FANCY, Marta Weigle. (22081-8) $1.50

Manufactured in the U.S.A.

CONTENTS

Chapter 1

FRIENDLY NUMBERS AND OTHERS

Any numerologist will tell you that numbers, **all** numbers, have personalities. However, you don't have to believe in numerology to believe in the personalities of **some** numbers. For example, how about the number 13? One immediately conjures up some thought connected with bad luck—an obnoxious personality has attached itself to this number 13. Numerically, 13 is just one of the smaller prime numbers (though it does have some special interest when we discuss Mersenne primes later).

Professional and amateur gamblers will tell you that the number 7 is schizophrenic—good luck at the right time, bad luck at the wrong time. The number 3 has had quite a reputation for being somewhat mystical—the Trinity, the 3 hours Christ hung on the Cross, the 3 days He lay in the tomb; Pythagoras called it the "perfect number" expressive of the "beginning, middle, and end" making it the symbol of Deity; Ancients considered the world ruled by 3 gods—Jupiter (in Heaven), Neptune (in the sea) and Pluto (in Hell); then there are the 3 Fates, the 3 Graces, the Muses which were 3 times 3; and Man is considered as body, soul and spirit. Similarly the number 7 has had a reputation for being mystical—the 7 days of creation, the 7 deadly sins, the ancient Hebrew "7 names of God." Here are two numbers, out of many, with definite religious associations.

From time immemorial the odd numbers have been endowed with male, the even numbers with female attributes, and *vice versa,* according to the culture concerned.

But these personalities associated with numbers are man-made and result from human experiences. There are numbers that have personalities inherent in the numbers themselves.

Some numbers are uninteresting in themselves[1] (except to a con-

[1] Notwithstanding the well-known argument which demonstrates that there are no uninteresting numbers! To wit: let us assume that there are two classes of numbers—interesting and uninteresting. Certainly all numbers could be put into

firmed numerologist) and acquire some status only when related to other numbers. The number 220 doesn't appear to have any special significance. But add up all its integer divisors, except the number itself:

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

Now do the same to the number 284:

$$1 + 2 + 4 + 71 + 142 = 220$$

We seem to have something here! 220 and 284 are intimately related since each is the sum of the divisors of the other. As a matter of fact, such number pairs are called *amicable* numbers, which means "friendly" numbers. Not too many of these number pairs are known—less than 1200—and the great Euler, in 1750, discovered 59 of them himself. Some other amicable number pairs are 1184 and 1210, 2620 and 2924, 5020 and 5564, 17,296 and 18,416, 9,363,584 and 9,437,056. The smaller pair, 220 and 284, has been known from antiquity and so much significance was attached to it that the possessor of one (in the form of talismans, or numerological significance) was assured [illegible]se friendship with the possessor of the other number of the pair. Undoubtedly, some marriages have been made on the basis of amicable numbers (considering the bases of some modern marriages, amicable number pairing may have had its points!).

The most studied class of numbers with definite inherent personalities is the class of prime numbers. Primes are integers which have no integer divisors, except themselves and 1: 2, 3, 5, 7, . . . 229, . . . 5693, . . . 199,999 and so on out to infinity. Prime numbers have rather defiant personalities and the larger the prime the more apparent is this defiance. Who cares that such a small number as 23 is prime? But how about 10,000,019 which has absolutely no integer divisors? Even this number is puny and insignificant compared to some of the LARGE primes which will be mentioned later.

Then there are some primes that look incredible:

---

one or the other of these classes. Now, in the class of uninteresting numbers there is a largest and a smallest number. Obviously, this makes them interesting! If we transfer these to the class of interesting numbers we will again have a largest and smallest uninteresting number, again making them interesting. Eventually we are left with one or two uninteresting numbers. But since they are the *only* uninteresting numbers, they are interesting for that reason! *Reductio ad absurdum.*

1,111,111,111,111,111,111
11,111,111,111,111,111,111,111
909,090,909,090,909,090,909,090,909,091
9,090,909,090,909,090,909,090,909,090,909,090,909,091

There are many and various methods available to test a given number to see if it is prime. For the very large primes special methods have been devised which, even then, must utilize electronic computers. Basically, however, the method involves dividing the given number by all the primes less than the square root of the given number. If we were to test 233 we would divide by all the primes less than $\sqrt{233}$, i.e., those primes less than 15. Dividing by 2, 3, 5, 7, 11 and 13 we find a remainder each time, so 233 is a prime. To test 5659 by this method we would have to divide by all the primes less than $\sqrt{5659}$, i.e. those less than 75. This involves 21 primes—though various tricks can obviate some of the divisions. But all the tricks in the trade won't make an easy task of determining the primality of a number like 8,083,457 which would involve division by the 412 primes less than $\sqrt{8,083,457}$. And who could ever use this method to check out the primality of those four incredible prime numbers shown above?

Prime numbers have been of major mathematical importance since Euclid, about 2300 years ago, showed that there was no limit to the number of primes. He argued roughly as follows: Suppose there were a largest prime P. The product of all the primes, plus 1, is either prime or non-prime (composite), i.e.,

$$(p_1 p_2 p_3 \ldots P) + 1 = \text{a prime or a composite number}$$

This number is not divisible by any of the primes smaller than P, or by P itself, since the remainder of 1 would result. Therefore, the number is either a prime, or it is a composite number with a prime factor larger than P. In either case, the existence of a prime greater than P is demanded. Therefore, there is no largest prime P—only a largest *known* prime.

Various tables of primes have been compiled. One of the most extensive of these is an 8-volume tabulation of the primes to 100,-330,201, which suffers from many errors (and its second volume has been lost). The original manuscript is preserved in Vienna and is still an extremely valuable document in spite of its inadequacies. Kulik, who spent much of his life on this table, listed over 5,761,456 primes. A nearly perfect table was published by Derrick N. Lehmer in 1914, and this includes all the primes up to and including 10,006,721—a list of 665,000 numbers including 1 (not usually considered a prime). Recently, microcard copies of the first 6,000,000 primes have been published. They were calculated by C. L. Baker and F. J. Gruenberger of the RAND Corporation on an IBM704 computer. This extensive table covers the primes up to 104,395,289.

Early Chinese manuscripts show that masculine qualities were ascribed to primes, although all other odd numbers were considered feminine. Even in those ancient days prime numbers were recognized as being very special, and now we have seen that modern mathematicians still feel that way about them.

In addition to the nearly 6,000,000 primes listed in the tables mentioned above, isolated larger primes of many types are known and most have some history attached to them—considering the tremendous task involved in establishing the primality of some of the larger primes. There is a group of prime numbers known as Robinson numbers given by the formula $R(k,n) = 2^n k + 1$ which yields primes for certain values of k and n. For $k = 5$ and $n = 1947$, the prime number formed contains 586 digits and is the largest known Robinson prime, though by no means the largest known prime.

Another formula which gives a few primes is one devised by Fermat, $2^{2^n} + 1$. Fermat firmly believed that this would yield primes for all values of n, but he was very wrong. Only five primes have been discovered which conform to this formula: 3, 5, 17, 257 and 65,537 from values of $n = 0, 1, 2, 3$ and 4 respectively. The very next value, $n = 5$, yields 4,294,967,297 which has the two prime factors 641 and 6,700,417. The compositeness of some Fermat numbers has been established, but no further primes have been discovered. The largest Fermat number tested is $2^{2^{1945}} + 1$ which contains about $10^{10^{584}}$ digits! Indeed, it is one of the largest numbers tested for primality. The complete factorization of this number is not known, but one of its prime factors is the large 586-digit Robinson prime $R(5, 1947)$. Since this large Fermat number, symbolized as F_{1945}, has been shown to be non-prime, it is doubtful that any further interest will be shown in establishing any of the other prime factors it may contain. There are limits to the attraction such numbers can have for even the most ardent number theorist.

The Fermat numbers have a close relationship with certain geometrical constructions, viz., the construction of regular polygons of n sides, where n is an odd prime. Although there is no difficulty in constructing equilateral triangles or regular pentagons, there are no methods available (using only a straightedge and compass) for constructing regular heptagons, or 11-gons, and others. A construction for a 17-gon is known and Carl Friedrich Gauss, in 1798 at about the age of 19, made the startling discovery that such constructions are possible if and only if n, the number of sides of the regular polygon, is a Fermat prime. In other words, Euclidean construction of a regular polygon with a prime number of sides is only possible if the number of sides is 3, 5, 17, 257 or 65,537.

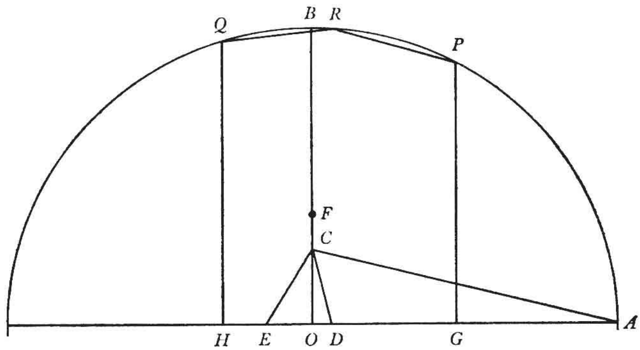


FIG. 1-1.

Equilateral triangles and regular pentagons are easy to draw. A regular 17-gon can be constructed as in Figure 1-1: [2]

Construct a semicircle with center at O and radius OA and draw a perpendicular OB. From A draw AC where OC is $\frac{1}{4}OB$. Construct $\angle OCD = \frac{1}{4}\ \angle OCA$ and $\angle ECD = 45°$. With EA as a diameter draw a semicircle which cuts OB in F. With D as center and DF as radius draw another semicircle and draw perpendiculars G and H where this semicircle cuts OA. P and Q, where the perpendiculars cut the large semicircle, form the extremities of an arc equal to 2/17 of the circumference. Bisection of this arc gives us R. PR or RQ is one side of the 17-gon.

[2] This is adapted from H. S. M. Coxeter, *Introduction to Geometry*, John Wiley & Sons, Inc., 1961, page 27.