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Preface

This book has expanded from our attempt to construct a general theory of hypergeometric functions and can be regarded as a first step towards its systematic exposition. However, this step turned out to be so interesting and important, and the whole program so overwhelming, that we decided to present it as a separate work. Moreover, in the process of writing we discovered a beautiful area which had been nearly forgotten so that our work can be regarded as a natural continuation of the classical developments in algebra during the 19th century.

We found that Cayley and other mathematicians of the period understood many of the concepts which today are commonly thought of as modern and quite recent. Thus, in an 1848 note on the resultant, Cayley in fact laid out the foundations of modern homological algebra. We were happy to enter into spiritual contact with this great mathematician.

The place of discriminants in the general theory of hypergeometric functions is similar to the place of quasi-classical approximation in quantum mechanics. More precisely, in [GGZ] [GKZ2] [GZK1] a general class of special functions was introduced and studied, the so-called A-hypergeometric functions. These functions satisfy a certain holonomic system of linear partial differential equations (the A-hypergeometric equations). The A-discriminant, which is one of our main objects of study, describes singularities of A-hypergeometric functions. According to the general principles of the theory of linear differential equations, these singularities are governed by the vanishing of the highest symbols of A-hypergeometric equations. The relation between differential operators and their highest symbols is the mathematical counterpart of the relation between quantum and classical mechanics; so we can say that hypergeometric functions provide a "quantization" of discriminants.

In our work on hypergeometric functions we found connections with many questions in algebra and combinatorics. We hope that this book brings to light some of these connections. One of the algebraic concepts which seems to us particularly important is that of hyperdeterminants (analogs of determinants for multi-dimensional "matrices.") After rediscovering hyperdeterminants in connection with hypergeometric functions, we found that they too, had been introduced by Cayley in the 1840s. Unfortunately, later on, the study of hyperdeterminants was largely abandoned in favor of another, more straightforward definition (cf. [P]). The only other work on hyperdeterminants of which we are aware is an important paper by Schläfli [Schl]. In this volume we give a detailed treatment of hyperdeterminants with the hope of attracting the attention of other mathematicians to this subject.

We would like to thank S.I. Gelfand, M.I. Graev and V.A. Vassiliev, who, through discussions and collaboration, have much influenced our understanding of the vast and beautiful field of hypergeometric functions.

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Introduction

I

In this book we study discriminants and resultants of polynomials in several variables. The most familiar example is the discriminant of a quadratic polynomial $f(x) = ax^2 + bx + c$. This is

$$\Delta(f) = b^2 - 4ac, \tag{1}$$

which vanishes when $f(x)$ has a double root.

More generally, we can consider a polynomial $f(x_1, \ldots, x_k)$ of degree $\leq d$ in k variables. An analog of a multiple root for f is a point where f vanishes together with all its first partial derivatives $\partial f / \partial x_i$. The *discriminant* $\Delta(f)$ is a polynomial function in the coefficients of f which vanishes whenever f has such a "multiple root." The existence of Δ is not quite trivial; however, it can be shown that $\Delta(f)$ exists and is unique up to sign if we require it to be irreducible and to have relatively prime integer coefficients. For instance, the discriminant of a cubic polynomial in one variable ($k = 1,\ d = 3$) is given by

$$\Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1^2 a_2^2 - 4a_1^3 a_3 - 4a_0 a_2^3 - 27a_0^2 a_3^2 + 18 a_0 a_1 a_2 a_3. \tag{2}$$

There is a subtle point in the definition of $\Delta(f)$: that is, $\Delta(f)$ depends not only on f but also on the choice of a degree bound d. For instance, the formula (2) applied to a quadratic polynomial gives a different expression from (1). With this in mind, we introduce the following more general version of a discriminant. Let A be a finite set of monomials in k variables, and let $\mathbf{C}^A$ denote the space of all polynomials with complex coefficients all of whose monomials belong to A. The *A-discriminant* $\Delta_A(f)$ is an irreducible polynomial in the coefficients of $f \in \mathbf{C}^A$ which vanishes whenever f has a multiple root $(x_1, \ldots, x_k)$ with all $x_i \neq 0$ (the last condition is added to be able to ignore trivial multiple roots which can appear if all monomials from A have high degree). The A-discriminant will be one of our main objects of study.

The notion of the A-discriminant includes as special cases several fundamental algebraic concepts. If we take $A = \{1, x, \ldots, x^m, y, yx, \ldots, yx^n\}$, for example, then a typical polynomial from $\mathbf{C}^A$ has the form $f(x) + yg(x)$. Its A-discriminant is the *resultant* of f and g: it vanishes whenever f and g have a common root.

More generally, the resultant of $k + 1$ polynomials $f_0, \ldots, f_k$ in k variables is defined as an irreducible polynomial in the coefficients of $f_0, \ldots, f_k$, which vanishes whenever these polynomials have a common root. The resultant can be treated as a special case of the A-discriminant of an auxiliary polynomial $f_0(x) + \sum_{i=1}^{k} y_i f_i(x)$, $x = (x_1, \ldots, x_k)$.

Another important example occurs when A consists of n^2 monomials $x_i y_j$, $i, j = 1, \ldots, n$. A typical polynomial from $\mathbf{C}^A$ is now a bilinear form $f(x, y) = \sum a_{ij} x_i y_j$ whose A-discriminant is the determinant of the matrix $\|a_{ij}\|$.

The last example has a natural generalization: we can take A as the set of all multilinear monomials in three or more groups of variables. An element $f \in \mathbf{C}^A$ (i.e., a multilinear form) is represented by a higher-dimensional "matrix" $\|a_{i_1 \ldots i_r}\|$. Thus the A-discriminant Δ_A in this case is a polynomial function of a "matrix" which extends the notion of a determinant. Following Cayley [Ca1], we call this Δ_A the *hyperdeterminant* of $\|a_{i_1 \ldots i_r}\|$. For example, the hyperdeterminant of a $2 \times 2 \times 2$ matrix $\|a_{ijk}\|$, $i, j, k = 0, 1$, is given by

$$(a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2)$$

$$-2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} + a_{001}a_{010}a_{101}a_{110}$$

$$+a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).$$

The study of hyperdeterminants was initiated by Cayley [Ca1] and Schläfli [Schl] but then was largely abandoned for 150 years. We present a treatment of hyperdeterminants in Chapter 14.

II

Let $\nabla_A = \{f \in \mathbf{C}^A : \Delta_A(f) = 0\}$ be the hypersurface in the space of polynomials consisting of polynomials with vanishing A-discriminant. We shall be mainly concerned with the following two closely related problems:

(a) the study of the geometric properties of the hypersurface ∇_A;

(b) finding an explicit algebraic expression of the discriminant Δ_A.

To illustrate the importance of problem (a), consider the special case when A consists of all monomials in $x_1, \ldots, x_k$ of a given degree d. Every $f \in \mathbf{C}^A$ (i.e., a homogeneous form of degree d) defines a hypersurface $\{f = 0\}$ in the projective space P^{k-1}. It is easy to see that ∇_A consists exactly of those f for which the hypersurface $\{f = 0\}$ is *singular*. Therefore the complement $\mathbf{C}^A - \nabla_A$ parametrizes all smooth hypersurfaces of a given degree in the projective space. To understand the geometric structure of $\mathbf{C}^A - \nabla_A$ is an important instance of the general moduli problem in algebraic geometry.

Equally important is the situation over the real numbers. Hilbert's 16th problem (classifying isotopy types of smooth real hypersurfaces of given degree d)

amounts to the study of connected components of $\mathbf{R}^A - \nabla_A$, the space of real polynomials with a non-vanishing discriminant.

Problem (b) has a long and glorious history. Explicit formulas for discriminants and resultants were the focus of several remarkable mathematicians in the last century. Many ingenious formulas were found by Cayley, Sylvester and their followers. However, we are still very far from a complete understanding of discriminants. For instance, an explicit polynomial expression for Δ_A is known only in a very limited number of special cases. Such formulas would be of great importance for the problem of finding explicit solutions of systems of polynomial equations. Problems of this kind are of interest not only for theoretical reasons, but are encountered more and more on a practical level because of the progress in computer technology.

III

We will use three main approaches in our study of discriminants and resultants:

- a geometric approach via projective duality and associated hypersurfaces;
- an algebraic approach via homological algebra and determinants of complexes (Whitehead torsion);
- a combinatorial approach via Newton polytopes and triangulations.

The geometric approach to discriminants is based on the observation that the discriminantal variety ∇_A is projectively dual to a certain variety X_A defined by a simple parametric representation. For example, if A consists of all monomials of degree d in k variables then X_A is the projective space P^{k-1} in its Veronese embedding. In the general case, X_A is the projective *toric* variety associated with A. The notion of the projectively dual variety $X^\vee$ makes sense for an arbitrary projective variety $X \subset P^{n-1}$: it is the closure of the set of all hyperplanes in P^{n-1} which are tangent to X at some smooth point. Thus the problem of finding the discriminant is a particular case of a more general geometric problem: find the equation(s) of $X^\vee$. We call this equation (in the case where $X^\vee$ is a hypersurface) the *X-discriminant.*

Although the resultants can be formally treated as discriminants of a special kind (see above), they have their own interesting geometric meaning. As for discriminants, we can associate the resultant to any projective variety $X \subset P^{n-1}$. Instead of $X^\vee$, we now consider the *associated hypersurface* $\mathcal{Z}(X)$ of X. If $\dim X = k-1$ then $\mathcal{Z}(X)$ is the locus of all codimension k projective subspaces in P^{n-1} which meet X. The equation of $\mathcal{Z}(X)$ in the appropriate Grassmannian is the classical *Chow form* of X. This can be represented as a polynomial in the coefficients of k linear forms defining a subspace from $\mathcal{Z}(X)$. We call this polynomial the *X-resultant* (the classical resultant of polynomials in several variables is a special case of this construction).

In Part I of this book we examine X-discriminants and X-resultants (or, in other words, projective duality and associated hypersurfaces) in the general context of projective geometry.

IV

The algebraic approach to discriminants and resultants which we use here goes back to Cayley. In his breathtaking 1848 note [Ca4] * he outlined a general method of writing down the resultant of several polynomials in several variables. We were very surprised to find that Cayley introduced in this note several fundamental concepts of homological algebra: complexes, exactness, Koszul complexes, and even the invariant now sometimes called the Whitehead torsion or Reidemeister-Franz torsion of an exact complex. The latter invariant is a natural generalization of the determinant of a square matrix (which itself was a rather recent discovery back in 1848!), so we prefer to call it the determinant of a complex. Using this terminology, Cayley's main result is that the resultant is the determinant of the Koszul complex.

Cayley's method is very general: without much effort it can be adapted to the study of X-discriminants and X-resultants associated as above to an arbitrary projective variety X. To get more detailed information, we complement Cayley's method with more recent tools such as coherent sheaves, perverse sheaves, microlocal geometry and D-modules.

V

Under a combinatorial approach we treat polynomials in the most naive way: as sums of monomials. To the best of our knowledge, there were no attempts in the classical literature to understand discriminants and resultants from this point of view, i.e., to describe which monomials can appear in them and with which coefficients. This is probably because the number of occurring monomials is usually very large. For example, the discriminant of a cubic form in three variables contains 2040 monomials (we are obliged to S. Duzhin who first showed it to us some years ago). At first glance, there seems to be no structure at all in these monomials and their coefficients. However, such a structure exists! The "magic crystal" that brings it to light is the concept of a *Newton polytope*.

Every monomial $x_1^{\omega_1} \cdots x_n^{\omega_n}$ in n variables can be visualized as a lattice point $(\omega_1, \ldots, \omega_n)$ in $\mathbf{R}^n$. The Newton polytope $N(F)$ of a polynomial $F(x_1, \ldots, x_n)$ is

* This note is reproduced as an appendix in this book

the convex hull in $\mathbf{R}^n$ of all lattice points representing monomials occurring in F. The structure of this polytope is deeply related to the geometry of the hypersurface $\{F = 0\}$. In fact, the asymptotic behavior of this hypersurface "at infinity" is controlled by the *extreme monomials* of F which correspond to the vertices of $N(F)$.

The notion of a Newton polytope goes back to Newton, and made some isolated appearances in the 19th century, cf. [Br 2]. More recently, some spectacular applications of Newton polytopes to classical algebraic problems (the number of solutions of systems of polynomial equations) have been found by A. Kouchnirenko, D. Bernstein, A. Khovansky [Ber], [Kou], [Kh]. We make use of these results in Part II.

It was a very surprising discovery for us when we realized that the Newton polytopes of A-discriminants admit a very nice combinatorial description. We recall that A is a finite set of monomials in k variables. As before, we represent the monomials from A as lattice points in $\mathbf{R}^k$. Hence we can consider the convex hull $Q \in \mathbf{R}^k$ of the set A. Our main result (which is the central point of Part II) is a description of the Newton polytope $N(\Delta_A)$ in terms of Q and A. Roughly speaking, it turns out that vertices of $N(\Delta_A)$ (i.e., extreme monomials in the A-discriminant) correspond to some *triangulations* of Q into simplices all of whose vertices lie in A.

The extreme monomial in Δ_A corresponding to a triangulation T of Q is determined explicitly once we know all the simplices in T and their volumes. The coefficient of this monomial is the product of numbers of the form $V_i^{V_i}$ where the V_i are the volumes of the simplices of T under suitable normalization. This provides an explanation of such coefficients as $4 = 2^2$ or $27 = 3^3$ in the formulas (1) and (2) above. The expression $\prod V_i^{V_i}$ (or, rather, its logarithm $\sum V_i \log(V_i)$) brings to mind the entropy of a probability distribution. It would be interesting to find a "probabilistic" reason for its appearance in discriminants. Even more intriguing is the fact that this appearance is not isolated—entropy-like expressions enter the formula for the rational uniformization of the variety ∇_A (see Chapter 9).

To illustrate the above description, consider the simplest A-discriminant of a quadratic polynomial $ax^2 + bx + c$ given by (1). Here A consists of $0, 1, 2 \in \mathbf{Z}$, the polytope Q is the segment $[0, 2]$, with its two "triangulations". The first one consists of just one 1-dimensional "simplex" $[0, 2]$ of length 2, corresponding to the term $-4ac$ in (1). The second "triangulation" consists of two "simplices": $[0, 1]$ and $[1, 2]$, corresponding to the term b^2. Similarly, for the case of a cubic polynomial in one variable, we have $A = \{0, 1, 2, 3\}$ and $Q = [0, 3]$. There are now 4 triangulations of Q which correspond to the first four terms in the discriminant (2). Our final example is the determinant of a 2×2 matrix given by a familiar formula $\Delta = ad - bc$. We have already seen that this is also a special case of an A-discriminant. The set A now consists of the vertices of a square Q; the terms ad and $-bc$ correspond to two triangulations of Q by means of one of its diagonals.

The description of the Newton polytope of Δ_A leads to a purely geometric notion of the *secondary polytope* $\Sigma(A)$ of a point configuration A. This is a polytope whose vertices correspond to the so-called *coherent* triangulations of the convex hull Q of A. Secondary polytopes and their generalizations (*fiber polytopes* introduced and studied by Billera and Sturmfels [BS1], [BS2]) are quite interesting by themselves. A triangulation of a polytope Q can be viewed as a discrete analog of a Riemannian metric on Q. So $\Sigma(A)$ can be seen as a kind of combinatorial Teichmüller space parametrizing such metrics. This reminds us of the work of Penner [Pen] who constructed a combinatorial model for the Teichmüller space of a Riemann surface in terms of its curvilinear triangulations.

VI

As mentioned in the Preface, our interest in the subject arose from the theory of hypergeometric functions [Ge] [GGZ] [GKZ2] [GZK1]. Although this theory is not formally present in the book, its influence is felt in several places. In a sense, one can say that hypergeometric functions provide a "quantization" of the discriminants. More precisely, to a finite set of monomials A, we associate a certain holonomic system of differential equations on the space $\mathbf{C}^A$ whose solutions are the so-called *A-hypergeometric functions*. The highest symbols of the equations of this system define, in the cotangent bundle of $\mathbf{C}^A$, the *characteristic variety* of the system. One of the components of this variety, when projected back to $\mathbf{C}^A$ is the discriminantal hypersurface ∇_A and the projections of other components are similar hypersurfaces associated to subsets of A.

The notion of a coherent triangulation, which plays such an essential part in our combinatorial approach to discriminants, was first brought to our attention by the analysis of A-hypergeometric functions. In fact, every coherent triangulation of the convex hull Q of A produces an explicit basis in the space of A-hypergeometric functions. This basis consists of a finite number of power series whose coefficients are products of the values of the Euler Γ-function.

VII

The book is subdivided into three parts. The first part is devoted to discriminants and resultants associated with arbitrary projective subvarieties. Most of the results here are classical but, to the best of our knowledge, have been never systematically treated in a book. Chapter 1 discusses projective duality. Chapter

2 introduces the Cayley method of expressing the discriminant as the determinant of a complex. Chapter 3 presents a parallel treatment of the resultants. Finally, Chapter 4 gives an exposition of the theory of Chow varieties (parameter spaces for projective subvarieties of given dimension and degree).

In Part II we consider A-discriminants and A-resultants. Geometrically, this corresponds to the specialization of the setting of Part I to projective toric varieties. We review toric varieties in Chapter 5 and the work of Bernstein and Kouchnirenko on Newton polytopes in Chapter 6. In Chapter 7 we present our main combinatorial-geometric construction: the secondary polytopes. In Chapters 8 – 11 this construction is related to Newton polytopes of A-discriminants and A-resultants. The main link between discriminants and triangulations is the so-called *principal A-determinant*. This is a certain product of discriminants whose Newton polytope is precisely the secondary polytope $\Sigma(A)$. For discriminants themselves, the correspondence between triangulations and the vertices of the Newton polytope is, in general, many-to-one.

Finally, Part III is devoted to the most classical examples of discriminants and resultants. The case of polynomials in one variable is treated in Chapter 12. Surprisingly, the point of view of Newton polytopes leads to new results even in this case. We treat the case of forms in several variables in Chapter 13, and hyperdeterminants in Chapter 14.

Geometrically, all of these examples correspond to varieties which are products of projective spaces $P^{l_1} \times \cdots \times P^{l_r}$ in a suitable projective embedding.

VIII

We did not attempt in this volume to collect all that is known about discriminants and resultants. The choice of material reflects both personal interests and the expertise of the authors.

Let us give a brief overview of some of the developments not included here but closely related to our subject. The following list is by no means complete.

First of all, an old tradition going back to Cayley and Sylvester, includes discriminants and resultants in the general context of the invariant theory of the group $GL(n)$. This approach involves expressing discriminants and resultants using the *symbolic method* (see e.g., [Go]). Our combinatorial approach focuses on the monomials, and thus is based on the action of the algebraic torus $(\mathbf{C}^*)^n$, not on the action of the whole group $GL(n)$.

Second, the study of discriminants and resultants constitutes only a part of Elimination Theory. There are other aspects of this theory which we do not discuss. Among those, we can mention the study of certain resultant ideals using