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Operator Theory, Analysis and Mathematical Physics

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Operator Theory: Advances and Applications
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Introduction

This volume contains mainly the lectures delivered by the participants of the International Conference: Operator Theory and its Applications in Mathematical Physics – OTAMP 2004, held at Mathematical Research and Conference Center in Bedlewo near Poznan. The idea behind these lectures was to present interesting ramifications of operator methods in current research of mathematical physics. The topics of these Proceedings are primarily concerned with: functional models of non-selfadjoint operators, spectral properties of Dirac and Jacobi matrices, Dirichlet-to-Neumann techniques, Lyapunov exponents methods and inverse spectral problems for quantum graphs.

All papers of the volume contain original material and were refereed by acknowledged experts.

The Editors thank all the referees whose critical remarks helped to improve the quality of this volume.

The Organizing Committee of the conference would like to thank all session organizers for taking care about the scientific programm and all participants for making warm and friendly atmosphere during the meeting.

We are particulary grateful to the organizers of **SPECT**, without whose financial support the OTAMP 2004 would never been so successful. We also acknowledge financial support of young Polish participants by Stefan Banach International Mathematical Center and thank the staff of the Conference Center at Bedlewo for their great support which helped to run the conference smoothly.

Finally, we thank the Editorial Board and especially Professor I. Gohberg for including this volume into the series **Operator Theory: Advances and Applications** and to Birkhäuser-Verlag for help in preparation of the volume.

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The Editors

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Finiteness of Eigenvalues of the Perturbed Dirac Operator

Petru A. Cojuhari

Abstract. Finiteness criteria are established for the point spectrum of the perturbed Dirac operator. The results are obtained by applying the direct methods of the perturbation theory of linear operators. The particular case of the Hamiltonian of a Dirac particle in an electromagnetic field is also considered.

Mathematics Subject Classification (2000). Primary 35P05, 47F05; Secondary 47A55, 47A75.

Keywords. Dirac operators, spectral theory, relatively compact perturbation.

1. Introduction

The present paper is concerned with a spectral problem for the perturbed Dirac operator of the form

$$H = \sum_{k=1}^{n} \alpha_k D_k + \alpha_{n+1} + Q, \tag{1.1}$$

where $D_k = i\frac{\partial}{\partial x_k}$ $(k = 1, \dots, n)$, α_k $(k = 1, \dots, n+1)$ are $m \times m$ Hermitian matrices which satisfy the anticommutation relations (or, so-called Clifford's relations)

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \ (j, k = 1, \dots, n+1), \tag{1.2}$$

$m = 2^{\frac{n}{2}}$ for n even and $m = 2^{\frac{n+1}{2}}$ for n odd. Q is considered as a perturbation of the free Dirac operator

$$H_0 = \sum_{k=1}^{n} \alpha_k D_k + \alpha_{n+1} \tag{1.3}$$

and represents the operator of multiplication by a given $m \times m$ Hermitian matrix-valued function $Q(x), x \in \mathbb{R}^n$. In accordance with our interests we assume that the elements $q_{jk}(x)$ $(j, k = 1, \dots, m)$ of the matrix $Q(x)$ are measurable functions from the space $L_\infty(\mathbb{R}^n)$. The operators H_0 and H are considered in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with their maximal domains of definition. Namely, it is considered that the domain of the operator H_0 is the Sobolev space $W_2^1(\mathbb{R}^n;\mathbb{C}^m)$ and, because Q is a bounded operator, the perturbed Dirac operator H is defined on the same domain $W_2^1(\mathbb{R}^n;\mathbb{C}^m)$ as well. The Dirac operators H_0 and H are selfadjoint on this domain. For the free Dirac operator H_0 is true the following algebraic relations

$$H_0^2 = \sum_{k=1}^{n} \alpha_k^2 D_k^2 + \sum_{j\neq k} (\alpha_j\alpha_k + \alpha_k\alpha_j) D_j D_k + \sum_{k=1}^{n} (\alpha_{n+1}\alpha_k + \alpha_k\alpha_{n+1}) D_k + \alpha_{n+1}^2$$
$$= \sum_{k=1}^{n} D_k^2 + E_m = (-\Delta + I)E_m,$$

so that

$$H_0^2 = (-\Delta + I)E_m. \tag{1.4}$$

Here Δ denotes the Laplace operator on $\mathbb{R}^n$ and E_m the $m\times m$ identity matrix. It follows from (1.4) that the spectrum of the operator H_0^2 covers the interval $[1,\infty]$ and, since the spectrum of the operator H_0 is a symmetric set with respect to the origin, it results that its spectrum coincides with the set $\sigma(H) = (-\infty,-1]\cup[1,+\infty)$. We note that the symmetry of the spectrum of H_0 can be shown easily by invoking, for instance, another matrix β which together with α_k $(k = 1,\ldots,n+1)$ the anticommutation conditions (1.2) are satisfied. Then

$$(H_0 + \lambda)\beta = -\beta(H_0 - \lambda)$$

for each scalar λ, and so the property of the symmetry of $\sigma(H_0)$ becomes to be clear. The unperturbed operator H_0 has no eigenvalues (in fact the spectrum of H_0 is only absolutely continuous). If the entries of the matrix-valued function $Q(x)$ vanish at the infinite, the continuous spectrum of the perturbed Dirac operator H coincides with $\sigma(H_0)$ and the perturbation Q can provoke a non-trivial point spectrum. Our problem is to study the point spectrum of the perturbed Dirac operator H. This problem has been studied by many researchers in connection with various problems (note that the most of the results were concerned with the case $n = 3$ and $m = 4$). A good deal of background material on the development and perspectives of the problem can be found in [1], [2], [3], [5], [7], [10], [12], [13], [14]. Apart from the already mentioned works, we refer to the [15] and the references given therein for a partial list.

In this paper, we give conditions on $Q(x)$ under which the point spectrum of H (if any) has ± 1 as the only possible accumulation points. Specifically, we assume that $Q(x)$ satisfies the following assumption.

(A) $Q(x) = [q_{jk}(x)]$, $x \in \mathbb{R}^n$, *is an* $m\times m$ *Hermitian matrix-valued function the entries of which are elements from the space* $L_\infty(\mathbb{R}^n)$ *and*

$$\lim_{|x|\to\infty} |\, x \,|\, q_{jk}(x) = 0 \ (j,k = 1,\ldots,m).$$

The main results are obtained by applying the abstract results from [6] (see also its refinement results made in [9]). Below, we cite the corresponding result.

Let $\mathcal{H}$ be a Hilbert space. Denote by $\mathbb{B}(\mathcal{H})$ the space of all bounded operators on $\mathcal{H}$ and by $\mathbb{B}_\infty(\mathcal{H})$ the subspace of $\mathbb{B}(\mathcal{H})$ consisting of all compact operators in $\mathcal{H}$. The domain and the range of an operator A are denoted by $\mathrm{Dom}(A)$ and $\mathrm{Ran}(A)$, respectively.

Theorem 1.1. [9] *Let A and B be symmetric operators in a space $\mathcal{H}$ and let the operator A has no eigenvalues on a closed interval Λ of the real axis. Suppose that there exists an operator-valued function $T(\lambda)$ defined on the interval Λ having the properties that*

(i) $T(\lambda) \in \mathbb{B}_\infty(\mathcal{H})$ $(\lambda \in \Lambda)$,

(ii) *$T(\lambda)$ is continuous on Λ in the uniform norm topology, and*

(iii) *for each $\lambda \in \Lambda$ and for each $u \in \mathrm{Dom}(B)$ such that $Bu \in \mathrm{Ran}(A - \lambda I)$ there holds the following inequality*

$$\| (A - \lambda I)^{-1} Bu \| \leq \| T(\lambda) u \| . \tag{1.5}$$

Then the point spectrum of the perturbed operator $A + B$ on the interval Λ consists only of finite number of eigenvalues of finity multiplicity.

Remark 1.2. *The assertion of Theorem* 1.1 *remains true if in place of* (1.5) *it is required the following one*

$$\| (A - \lambda I)^{-1} Bu \| \leq \sum_{k=1}^{N} \| T_k(\lambda) u \|, \tag{1.6}$$

where the operator-valued functions $T_k(\lambda)$ $(k = 1, \ldots, N)$ satisfy the conditions (i) *and* (ii).

As we already mentioned we will apply Theorem 1.1 to the study of the problem of the discreteness of the set of eigenvalues of the perturbed Dirac operator H. The main results are presented in the next section.

2. Main results

Let H be the Dirac operator defined by (1.1) in which the matrix-valued function satisfies the assumption (A). The unperturbed Dirac operator H_0 represents a matrix differential operator (of the dimension $m \times m$) of order 1. The symbol of the operator H_0 is a matrix-valued function which we denote by $h_0(\xi), \xi \in \mathbb{R}^n$. Note that by applying the Fourier transformation to the elements of the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ the operator H_0 is transformed (in the momentum space) into a multiplication operator by the matrix $h_0(\xi)$. The Fourier transformation is defined by the formula

$$\hat{u}(\xi) = (Fu)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int u(x) e^{i<x,\xi>} dx \ (u \in L_2(\mathbb{R}^n))$$

in which $< x, \xi >$ designates the scalar product of the elements $x, \xi \in \mathbb{R}^n$ (here and in what follows $\int : \int_{\mathbb{R}^n}$). The corresponding norm in $\mathbb{R}^n$ (or $\mathbb{C}^m$) will be denoted as usually by $| \, . \, |$. The operator norm of $m \times m$ matrices corresponding to the norm $| \, . \, |$ in $\mathbb{C}^m$ will be denoted by $| \, . \, |$, as well.

Our main result is the following

Theorem 2.1. *Let H be the perturbed Dirac operator defined by* (1.1) *for which the assumption* (A) *is satisfied. Then the point spectrum of the operator H has only ± 1 as accumulation points. Each eigenvalue can be only of a finite multiplicity.*

Proof. That the spectrum in the spectral gap $(-1, 1)$ is only discrete without any accumulation points in the interior of this interval follows at once due to Weyl type theorems. Let Λ be an closed interval contained in the set $(-\infty, -1) \cup (1, +\infty)$ and let λ be an arbitrary point belonging to Λ. It will be shown that under assumed conditions the operators H_0 and H verify all of hypotheses of Theorem 1.1. To this end, we estimate the norm of the element $(H_0 - \lambda I)^{-1} Qu$ for each $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ such that $Qu \in \operatorname{Ran}(H_0 - \lambda I)$. Let $\widehat{Qu}$ be the Fourier transform of Qu, and denote

$$\widehat{v}(\xi) := (h_0(\xi) + \lambda)\widehat{Qu}(\xi), \ \xi \in \mathbb{R}^n.$$

According to (1.4), we may write

$$\begin{aligned} \| \, (H_0 - \lambda I)^{-1} Qu \, \|^2 &= \int |(h_0(\xi) - \lambda)^{-1} \widehat{Qu}(\xi)|^2 d\xi \\ &= \int |(|\xi|^2 - r(\lambda)^2)^{-1} \hat{v}(\xi)|^2 d\xi, \end{aligned} \tag{2.1}$$

where $r(\lambda) := \sqrt{\lambda^2 - 1}$.

Next, we let

$$\Omega(\Lambda) = \cup_{\lambda \in \Lambda} \{\xi \in \mathbb{R}^n : | \, \xi \, | = r(\lambda)\}$$

and we choose a sphere U of radius R with center of the origin such that $U \supset \Omega(\Lambda)$ and let $V = \mathbb{R}^n \setminus U$. Then passing to spherical coordinates $\xi = |\xi|\omega, \rho = |\xi|$ (we write ds_ω for the area element of hypersurface S_{n-1} of the unit sphere S in $\mathbb{R}^n$), and denoting

$$\hat{f}(\rho, \omega) = \frac{\rho^{\frac{n-1}{2}} \hat{v}(\rho\omega)}{\rho + r(\lambda)} \ (0 \le \rho < \infty, \ \omega \in S_{n-1}),$$

we have

$$\begin{aligned} \int_U |(h_0(\xi) - \lambda)^{-1} \widehat{Qu}(\xi)|^2 d\xi &= \int_{S_{n-1}} \int_0^R \rho^{n-1} |(\rho^2 - r(\lambda)^2)^{-1} \hat{v}(\rho\omega)|^2 d\rho dS_\omega \\ &= \int_{S_{n-1}} \int_0^R \left| \frac{\hat{f}(\rho, \omega)}{\rho - r(\lambda)} \right|^2 d\rho dS_\omega. \end{aligned}$$

Since $Qu \in \operatorname{Ran}(H_0 - \lambda I)$, it follows that $\hat{f}(\rho, \omega)$ vanishes at $\rho = r(\lambda)$, and we can continue

$$\begin{aligned} &\left[\int_{S_{n-1}} \int_0^R \left| \frac{\hat{f}(\rho, \omega)}{\rho - r(\lambda)} \right|^2 d\rho dS_\omega \right]^{\frac{1}{2}} \\ &= \left[\int_{S_{n-1}} \int_0^R \left| \int_0^1 \frac{\partial \hat{f}}{\partial \rho}(t(\rho - r(\lambda)) + r(\lambda), \omega) dt \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}} \end{aligned}$$

$$\leq \int_0^1 \left[\int_{S_{n-1}} \int_0^R \left| \frac{\partial \hat{f}}{\partial \rho}(t(\rho - r(\lambda)) + r(\lambda), \omega)|^2 d\rho ds_\omega \right]^{\frac{1}{2}} dt$$

$$\leq 2\left[\int_{S_{n-1}} \int_0^R \left| \frac{\partial \hat{f}}{\partial \rho}(\rho, \omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}}$$

$$\leq \left[\int_{S_{n-1}} \int_0^R \left| \rho^{\frac{n-3}{2}} ((n-3)\rho + (n-1)r(\lambda))(\rho + r(\lambda))^{-2} \hat{v}(\rho\omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}}$$

$$+ 2\left[\int_{S_{n-1}} \int_0^R \left| \rho^{\frac{n-1}{2}} (\rho + r(\lambda))^{-1} \frac{\partial}{\partial \rho} \hat{v}(\rho\omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}}.$$

Taking into account that $|\frac{\partial}{\partial \rho}\hat{v}(\rho\omega)| \leq |\nabla \hat{v}|$, we get

$$\left[\int_U \left| (h_0(\xi) - \lambda)^{-1} \widehat{Qu}(\xi) \right|^2 d\xi \right]^2$$

$$\leq 2r(\lambda) \left[\int_U \left| \frac{(n-3) \mid \xi \mid + (n-1)r(\lambda)}{|\xi|(|\xi| + r(\lambda))^2} \hat{v}(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} + 2\left[\int_U \left| \frac{\nabla \hat{v}(\xi)}{|\xi| + r(\lambda)} \right|^2 d\xi \right]^{\frac{1}{2}}.$$

Since the expressions $(n-3) \mid \xi \mid + (n-1)r(\lambda)$, $(\mid \xi \mid + r(\lambda))^{-1}$ $(\lambda \in \Lambda; \xi \in U)$ and each element of the matrix-valued function $h_0(\xi) - \lambda$ $(\lambda \in \Lambda; \xi \in U)$ are bounded on $\Lambda \times U$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\left[\int_U |(h_0(\xi) - \lambda)^{-1} \widehat{Qu}(\xi) \right|^2 d\xi \right]^{\frac{1}{2}}$$

$$\leq c_1 \left[\int_U \left| \mid \xi \mid^{-1} \widehat{Qu}(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} + c_2 \left[\int_U |\nabla \widehat{Qu}(\xi)|^2 d\xi \right]^{\frac{1}{2}}.$$

We claim that the integral operators with kernels

$$\mid \xi \mid^{-1} Q(x) e^{-i(x,\xi)}, x^l Q(x) e^{-i(x,\xi)} \ (\mid l \mid = 1; x \in \mathbb{R}^n, \xi \in U)$$

are compact operators in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$. The compactness of them can be proved by applying the criteria obtained in [4] (or, also, by applying the lemma from [8], page 45).

In addition, we note that the integral operator K_V with the kernel

$$(h_0(\xi) - \lambda)^{-1} Q(x) e^{-i(x,\xi)} \ (x \in \mathbb{R}^n; \xi \in V)$$

represents also a compact operator. To see this fact, it suffices to show that

$$\| (I - P_h)K_V \| \to 0 \ as \ h \to \infty, \tag{2.2}$$

where $(P_h u)(x) = u(x)$ for $\mid x \mid \leq h$ and $(P_h u)(x) = 0$ for $\mid x \mid > h$.

Since each element of the matrix-valued function $(h_0(\xi) - \lambda)^{-1}$ behaves as $\mid \xi \mid^{-1}$ at the infinite, it follows the evaluation

$$\| (I - P_h)K_V u \|^2 \leq c \int_{|\xi| > h} |(1 + \mid \xi \mid)^{-1} \widehat{Qu}(\xi)|^2 d\xi \leq c(1+h)^{-2} \| u \|^2,$$

and so (2.2) is realized.

Thus, taking into account (2.1), we obtain an estimate like that from (1.6) (see Remark 1.2) and, therefore Theorem 1.1 can be applied. This completes the proof of Theorem 2.1. □

As an application of Theorem 2.1 we give a result concerning the particular case of the Hamiltonian of a Dirac particle in an electromagnetic field. The Dirac operator in this case is typically written in the physics literature (see, for instance, [11], [15]) as follows

$$Hu = \sum_{j=1}^{3} \alpha_j (D_j - A_j(x))u + \alpha_4 u + q(x)u, u \in W_2^1(\mathbb{R}^3; \mathbb{C}^4), \qquad (2.3)$$

where $A(x) = (A_1(x), A_2(x), A_3(x))$ (the vector potential) and $q(x)$ (the scalar potential) are given functions on $\mathbb{R}^3$.

Theorem 2.2. *If*

$$\lim_{|x|\to\infty} |x|A_j(x) = 0 \ (j = 1, 2, 3), \lim_{|x|\to\infty} |x|q(x) = 0,$$

then the point spectrum of the Dirac operator defined by (2.1) *is discrete having only* ± 1 *as accumulation points. Each eigenvalue can be only of a finite multiplicity.*

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