

**Operator Theory
Advances and Applications
Vol. 174**

Operator Theory, Analysis and Mathematical Physics

**Jan Janas
Pavel Kurasov
Ari Laptev
Sergei Nabako
Günter Stolz
Editors**

Birkhäuser

0177-53

061

2004

Operator Theory, Analysis and Mathematical Physics

Jan Janas
Pavel Kurasov
Ari Laptev
Sergei Naboko
Günter Stolz
Editors



E2007002204

Birkhäuser
Basel · Boston · Berlin

Editors:

Jan Janas
Institut of Mathematics
Polish Academy of Sciences
ul. Sw. Tomasza 30/7
31-027 Krakow
Poland
e-mail: najanas@cyf-kr.edu.pl

Pavel Kurasov
Department of Mathematics
Lund Institute of Technology
221 00 Lund
Sweden
e-mail: kurasov@maths.lth.se

Sergei Naboko
Department of Mathematical Physics
St. Petersburg State University
Ulyanovskaya 1
198904 St. Petersburg
Russia
e-mail: naboko@snoopy.phys.spbu.ru

Ari Laptev
Dept. of Mathematics, Room 675
South Kensington Campus
Imperial College London
SW7 2AZ London, UK
e-mail: a.laptev@imperial.ac.uk

and

Department of Mathematics
Royal Institute of Technology
100 44 Stockholm
Sweden
e-mail: laptev@math.kth.se

Günter Stolz
Department of Mathematics
University of Alabama
452 Campbell Hall
Birmingham, AL 35294-1170
USA
e-mail: stolz@math.uab.edu

2000 Mathematics Subject Classification 35J10, 35P05, 37K15, 39A11, 39A12, 47A10, 47A40, 47A45, 47A48, 47B36, 47F05, 81V99, 82D10, 99Z99

Library of Congress Control Number: 2007922252

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

ISBN 978-3-7643-8134-9 Birkhäuser Verlag, Basel – Boston – Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in other ways, and storage in data banks. For any kind of use permission of the copyright owner must be obtained.

© 2007 Birkhäuser Verlag AG, P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media

Printed on acid-free paper produced from chlorine-free pulp. TCF ∞

Cover design: Heinz Hiltbrunner, Basel

Printed in Germany

ISBN-10: 3-7643-8134-5

ISBN-13: 978-3-7643-8134-9

9 8 7 6 5 4 3 2 1

e-ISBN-10: 3-7643-8135-3

e-ISBN-13: 978-3-7643-8135-6

www.birkhauser.ch



Operator Theory: Advances and Applications
Vol. 174

Editor:
I. Gohberg

Editorial Office:
School of Mathematical
Sciences
Tel Aviv University
Ramat Aviv, Israel

Editorial Board:
D. Alpay (Beer-Sheva)
J. Arazy (Haifa)
A. Atzmon (Tel Aviv)
J. A. Ball (Blacksburg)
A. Ben-Artzi (Tel Aviv)
H. Bercovici (Bloomington)
A. Böttcher (Chemnitz)
K. Clancey (Athens, USA)
L. A. Coburn (Buffalo)
R. E. Curto (Iowa City)
K. R. Davidson (Waterloo, Ontario)
R. G. Douglas (College Station)
A. Dijksma (Groningen)
H. Dym (Rehovot)
P. A. Fuhrmann (Beer Sheva)
B. Gramsch (Mainz)
J. A. Helton (La Jolla)
M. A. Kaashoek (Amsterdam)
H. G. Kaper (Argonne)

S. T. Kuroda (Tokyo)
P. Lancaster (Calgary)
L. E. Lerer (Haifa)
B. Mityagin (Columbus)
V. Olshevsky (Storrs)
M. Putinar (Santa Barbara)
L. Rodman (Williamsburg)
J. Rovnyak (Charlottesville)
D. E. Sarason (Berkeley)
I. M. Spitkovsky (Williamsburg)
S. Treil (Providence)
H. Upmeyer (Marburg)
S. M. Verduyn Lunel (Leiden)
D. Voiculescu (Berkeley)
D. Xia (Nashville)
D. Yafaev (Rennes)

Honorary and Advisory
Editorial Board:
C. Foias (Bloomington)
P. R. Halmos (Santa Clara)
T. Kailath (Stanford)
H. Langer (Vienna)
P. D. Lax (New York)
M. S. Livsic (Beer Sheva)
H. Widom (Santa Cruz)

Introduction

This volume contains mainly the lectures delivered by the participants of the International Conference: Operator Theory and its Applications in Mathematical Physics – OTAMP 2004, held at Mathematical Research and Conference Center in Bedlewo near Poznan. The idea behind these lectures was to present interesting ramifications of operator methods in current research of mathematical physics. The topics of these Proceedings are primarily concerned with: functional models of non-selfadjoint operators, spectral properties of Dirac and Jacobi matrices, Dirichlet-to-Neumann techniques, Lyapunov exponents methods and inverse spectral problems for quantum graphs.

All papers of the volume contain original material and were refereed by acknowledged experts.

The Editors thank all the referees whose critical remarks helped to improve the quality of this volume.

The Organizing Committee of the conference would like to thank all session organizers for taking care about the scientific program and all participants for making warm and friendly atmosphere during the meeting.

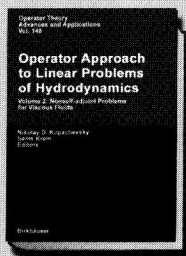
We are particularly grateful to the organizers of **SPECT**, without whose financial support the OTAMP 2004 would never been so successful. We also acknowledge financial support of young Polish participants by Stefan Banach International Mathematical Center and thank the staff of the Conference Center at Bedlewo for their great support which helped to run the conference smoothly.

Finally, we thank the Editorial Board and especially Professor I. Gohberg for including this volume into the series **Operator Theory: Advances and Applications** and to Birkhäuser-Verlag for help in preparation of the volume.

Birmingham – Krakow – Lund
St. Petersburg – Stockholm

August 2006
The Editors

Operator Theory: Advances and Applications



Edited by
Gohberg, I., School of Mathematical Sciences, Tel Aviv University,
Ramat Aviv, Israel

This series is devoted to the publication of current research in operator theory, with particular emphasis on applications to classical analysis and the theory of integral equations, as well as to numerical analysis, mathematical physics and mathematical methods in electrical engineering.

**Your Specialized
Publisher in
Mathematics
Birkhäuser**



For orders originating from all over the world
except USA/Canada/Latin America:

Birkhäuser Verlag AG
c/o Springer GmbH & Co
Haberstrasse 7
D-69126 Heidelberg
Fax: +49 / 6221 / 345 4 229
e-mail: birkhauser@springer.de
http://www.birkhauser.ch

For orders originating in the
USA/Canada/Latin America:

Birkhäuser
333 Meadowland Parkway
USA-Secaucus
NJ 07094-2491
Fax: +1 201 348 4505
e-mail: orders@birkhauser.com

OT 176: Alpay, D. / Vinnikov, V. (Eds.), System Theory, the Schur Algorithm and Multidimensional Analysis (2007). ISBN 978-3-7643-8136-3

OT 175: Förster, K.-H. / Jonas, P. / Langer, H. / Trunk, C. (Eds.), Operator Theory in Inner Product Spaces (2007). ISBN 978-3-7643-8269-8

OT 174: Janas, J. / Kurasov, P. / Laptev, A. / Naboko, S. / Stolz, G. (Eds.), Operator Theory, Analysis and Mathematical Physics (2007). ISBN 978-3-7643-8134-9

OT 173: Emel'yanov, E.Yu., Non-spectral Asymptotic Analysis of One-Parameter Operator Semigroups (2006). ISBN 978-3-7643-8095-3

OT 172: Toft, J. / Wong, M.W. / Zhu, H. (Eds.), Modern Trends in Pseudo-Differential Operators (2007). ISBN 3-7643-8097-7

OT 171: Dritschel, M. (Ed.), The Extended Field of Operator Theory (2007). ISBN 3-7643-7979-0

OT 170: Erusalimsky, Ya.M. / Gohberg, I. / Grudsky, S.M. / Rabinovich, V. / Vasilevski, N. (Eds.), Modern Operator Theory and Applications. The Igor Borisovich Simonenko Anniversary Volume (2007). ISBN 3-7643-7736-4

OT 169: Haase, M., The Functional Calculus for Sectorial Operators (2006). ISBN 3-7643-7697-X

OT 168: Koelink, E. / van Neerven, J. / de Pagter, B. / Sweers, G. (Eds.), Partial Differential Equations and Functional Analysis. The Philippe Clément Festschrift (2006). ISBN 3-7643-7600-7

OT 167: Alpay, D. (Ed.), Wavelets, Multiscale Systems and Hypercomplex Analysis (2006). ISBN 3-7643-7587-6

OT 166: De Gosson, M., Symplectic Geometry and Quantum Mechanics (2006). Subseries **Advances in Partial Differential Equations** ISBN 3-7643-7574-4

OT 165: Alpay, D. / Gohberg, I. (Eds.), Interpolation, Schur Functions and Moment Problems (2006). Subseries **Linear Operators and Linear Systems** ISBN 3-7643-7546-9

OT 164: Boggiatto, P. / Rodino, L. / Toft, J. / Wong, M.W. (Eds.), Pseudo-Differential Operators and Related Topics (2006). ISBN 3-7643-7513-2

OT 163: Langer, M. / Luger, A. / Woracek, H. (Eds.), Operator Theory and Indefinite Inner Product Spaces (2006). ISBN 3-7643-7515-9

Operator Theory: Advances and Applications

Your Specialized
Publisher in
Mathematics
Birkhäuser



OT 162: Förster, K.-H. / Jonas, P. / Langer, H. (Eds.), *Operator Theory in Krein Spaces and Nonlinear Eigenvalue Problems* (2006). ISBN 3-7643-7452-7

OT 161: Alpay, D. / Gohberg, I. (Eds.), *The State Space Method. Generalizations and Applications* (2005). Subseries *Linear Operators and Linear Systems* ISBN 3-7643-7370-9

OT 160: Kaashoek, M.A. / Seatzu, S. / van der Mee, C. (Eds.), *Recent Advances in Operator Theory and its Applications. The Israel Gohberg Anniversary Volume* (2005). ISBN 3-7643-7290-7

OT 159: Reissig, M. / Schulze, B.-W. (Eds.), *New Trends in the Theory of Hyperbolic Functions* (2005). Subseries *Advances in Partial Differential Equations* ISBN 3-7643-7283-4

OT 158: Eiderman, V.Ya. / Samokhin, M.V. (Eds.), *Selected Topics in Complex Analysis* (2005). ISBN 3-7643-7251-6

OT 157: Alpay, D. / Vinnikov, V. (Eds.), *Operator Theory, Systems Theory and Scattering Theory: Multidimensional Generalizations* (2005). ISBN 3-7643-7212-5

OT 156: Ebenfelt, P. / Gustafsson, B. / Khavinson, D. / Putinar, M. (Eds.), *Quadrature Domains and Their Applications. The Harold S. Shapiro Anniversary Volume* (2005). ISBN 3-7643-7145-5

OT 155: Ashino, R. / Boggiatto, P. / Wong, M.W. (Eds.), *Advances in Pseudo-Differential Operators* (2004). ISBN 3-7643-7140-4

OT 154: Janas, J. / Kurasov, P. / Naboko, S. (Eds.), *Spectral Methods for Operators of Mathematical Physics* (2004). ISBN 3-7643-7133-1

OT 153: Gaspar, D. / Gohberg, I. / Timotin, D. / Vasilescu, F.H. / Zsido, L. (Eds.), *Recent Advances in Operator Theory, Operator Algebras, and their Applications* (2004). ISBN 3-7643-7127-7

OT 152: Eidelman, S.D. / Ivasyshen, S.D. / Kochubei, A.N., *Analytic Methods in the Theory of Differential and Pseudo-differential Equations of Parabolic Type* (2004). ISBN 3-7643-7115-3

OT 151: Gil, J.B. / Krainer, T. / Witt, I. (Eds.), *Aspects of Boundary Problems in Analysis and Geometry* (2004). Subseries *Advances in Partial Differential Equations* ISBN 3-7643-7069-6

OT 150: Rabinovich, V. / Roch, S. / Silbermann, B., *Limit Operators and their Applications in Operator Theory* (2004). ISBN 3-7643-7081-5

OT 149: Ball, J.A. / Helton, J.W. / Klaus, M. / Rodman, L. (Eds.), *Current Trends in Operator Theory and its Applications* (2004). ISBN 3-7643-7067-X

OT 148: Ashyralyev, A. / Sobolevskii, P.E., *New Difference Schemes for Partial Differential Equations* (2004). ISBN 3-7643-7054-8

OT 147: Gohberg, I. / dos Santos, A.F. / Speck, F.-O. / Teixeira, F.S. / Wendland, W. (Eds.), *Operator Theoretical Methods and Applications to Mathematical Physics. The Erhard Meister Memorial Volume* (2004). ISBN 3-7643-6634-6

OT 146: Kopachevsky, N.D. / Krein, S.G., *Operator Approach to Linear Problems of Hydrodynamics. Volume 2: Nonself-adjoint Problems for Viscous Fluids* (2003). ISBN 3-7643-2190-3

OT 145: Albeverio, S. / Demuth, M. / Schrohe, E. / Schulze, B.-W. (Eds.), *Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet Transformations* (2003). Subseries *Advances in Partial Differential Equations* ISBN 3-7643-2168-7

OT 144: Belitskii, G. / Tkachenko, V., *One-dimensional Functional Equations* (2003). ISBN 3-7643-0084-1

OT 143: Alpay, D. (Ed.), *Reproducing Kernel Spaces and Applications* (2003). ISBN 3-7643-0068-X

OT 142: Böttcher, A. / dos Santos, A.F. / Kaashoek, M.A. / Brites Lebre, A. / Speck, F.-O. (Eds.), *Singular Integral Operators, Factorization and Applications* (2003). ISBN 3-7643-6947-7

OT 141: dos Santos, A.F. / Gohberg, I. / Manojlovic, N. (Eds.), *Factorization and Integrable Systems. Proceedings of the Summer School, Faro, Portugal, 2000* (2003). ISBN 3-7643-6938-8

OT 140: Ellis, R. / Gohberg, I., *Orthogonal Systems and Convolution Operators* (2002). ISBN 3-7643-6929-9

OT 139: Müller, V., *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras* (2003). ISBN 3-7643-6912-4

OT 138: Albeverio, S. / Demuth, M. / Schrohe, E. / Schulze, B.-W. (Eds.), *Parabolicity, Volterra Calculus, and Conical Singularities* (2002). Subseries *Advances in Partial Differential Equations*. ISBN 3-7643-6906-X

OT 137: Dybin, V. / Grudsky, S.M., *Introduction to the Theory of Toeplitz Operators with Infinite Index* (2002). ISBN 3-7643-6906-X

OT 136: Wong, M.W., *Wavelet Transforms and Localization Operators* (2002). ISBN 3-7643-6789-X

OT 135: Böttcher, A. / Gohberg, I. / Junghanns, P. (Eds.), *Toeplitz Matrices, Convolution Operators, and Integral Equations. The Bernd Silbermann Anniversary Volume* (2002). ISBN 3-7643-6877-2

OT 134: Alpay, D. / Gohberg, I. / Vinnikov, V. (Eds.), *Interpolation Theory, Systems Theory and Related Topics. The Harry Dym Anniversary Volume* (2002). ISBN 3-7643-6762-8

Contents

Preface	vii
<i>P.A. Cojuhari</i> Finiteness of Eigenvalues of the Perturbed Dirac Operator	1
<i>M. Combescure</i> A Mathematical Study of Quantum Revivals and Quantum Fidelity	9
<i>P. Exner, T. Ichinose and S. Kondej</i> On Relations Between Stable and Zeno Dynamics in a Leaky Graph Decay Model	21
<i>R.L. Frank and R.G. Shterenberg</i> On the Spectrum of Partially Periodic Operators	35
<i>A.V. Kiselev</i> Functional Model for Singular Perturbations of Non-self-adjoint Operators	51
<i>J. Michor and G. Teschl</i> Trace Formulas for Jacobi Operators in Connection with Scattering Theory for Quasi-Periodic Background	69
<i>A.B. Mikhailova, B.S. Pavlov and V.I. Ryzhii</i> Dirichlet-to Neumann Techniques for the Plasma-waves in a Slot-diod	77
<i>M. Nowaczyk</i> Inverse Spectral Problem for Quantum Graphs with Rationally Dependent Edges	105
<i>V. Ryzhov</i> Functional Model of a Class of Non-selfadjoint Extensions of Symmetric Operators	117
<i>H. Schulz-Baldes</i> Lyapunov Exponents at Anomalies of $SL(2, \mathbb{R})$ -actions	159

<i>L.O. Silva</i>	
Uniform and Smooth Benzaid-Lutz Type Theorems and Applications to Jacobi Matrices	173
<i>S. Simonov</i>	
An Example of Spectral Phase Transition Phenomenon in a Class of Jacobi Matrices with Periodically Modulated Weights	187
<i>A. Tikhonov</i>	
On Connection Between Factorizations of Weighted Schur Function and Invariant Subspaces	205
List of Participants in OTAMP2004	247

Finiteness of Eigenvalues of the Perturbed Dirac Operator

Petru A. Cojuhari

Abstract. Finiteness criteria are established for the point spectrum of the perturbed Dirac operator. The results are obtained by applying the direct methods of the perturbation theory of linear operators. The particular case of the Hamiltonian of a Dirac particle in an electromagnetic field is also considered.

Mathematics Subject Classification (2000). Primary 35P05, 47F05; Secondary 47A55, 47A75.

Keywords. Dirac operators, spectral theory, relatively compact perturbation.

1. Introduction

The present paper is concerned with a spectral problem for the perturbed Dirac operator of the form

$$H = \sum_{k=1}^n \alpha_k D_k + \alpha_{n+1} + Q, \quad (1.1)$$

where $D_k = i \frac{\partial}{\partial x_k}$ ($k = 1, \dots, n$), α_k ($k = 1, \dots, n+1$) are $m \times m$ Hermitian matrices which satisfy the anticommutation relations (or, so-called Clifford's relations)

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \quad (j, k = 1, \dots, n+1), \quad (1.2)$$

$m = 2^{\frac{n}{2}}$ for n even and $m = 2^{\frac{n+1}{2}}$ for n odd. Q is considered as a perturbation of the free Dirac operator

$$H_0 = \sum_{k=1}^n \alpha_k D_k + \alpha_{n+1} \quad (1.3)$$

and represents the operator of multiplication by a given $m \times m$ Hermitian matrix-valued function $Q(x)$, $x \in \mathbb{R}^n$. In accordance with our interests we assume that the elements $q_{jk}(x)$ ($j, k = 1, \dots, m$) of the matrix $Q(x)$ are measurable functions from the space $L_\infty(\mathbb{R}^n)$. The operators H_0 and H are considered in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with their maximal domains of definition. Namely, it is considered

that the domain of the operator H_0 is the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ and, because Q is a bounded operator, the perturbed Dirac operator H is defined on the same domain $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ as well. The Dirac operators H_0 and H are selfadjoint on this domain. For the free Dirac operator H_0 is true the following algebraic relations

$$\begin{aligned} H_0^2 &= \sum_{k=1}^n \alpha_k^2 D_k^2 + \sum_{j \neq k} (\alpha_j \alpha_k + \alpha_k \alpha_j) D_j D_k + \sum_{k=1}^n (\alpha_{n+1} \alpha_k + \alpha_k \alpha_{n+1}) D_k + \alpha_{n+1}^2 \\ &= \sum_{k=1}^n D_k^2 + E_m = (-\Delta + I) E_m, \end{aligned}$$

so that

$$H_0^2 = (-\Delta + I) E_m. \quad (1.4)$$

Here Δ denotes the Laplace operator on \mathbb{R}^n and E_m the $m \times m$ identity matrix. It follows from (1.4) that the spectrum of the operator H_0^2 covers the interval $[1, \infty]$ and, since the spectrum of the operator H_0 is a symmetric set with respect to the origin, it results that its spectrum coincides with the set $\sigma(H) = (-\infty, -1] \cup [1, +\infty)$. We note that the symmetry of the spectrum of H_0 can be shown easily by invoking, for instance, another matrix β which together with α_k ($k = 1, \dots, n+1$) the anticommutation conditions (1.2) are satisfied. Then

$$(H_0 + \lambda)\beta = -\beta(H_0 - \lambda)$$

for each scalar λ , and so the property of the symmetry of $\sigma(H_0)$ becomes to be clear. The unperturbed operator H_0 has no eigenvalues (in fact the spectrum of H_0 is only absolutely continuous). If the entries of the matrix-valued function $Q(x)$ vanish at the infinite, the continuous spectrum of the perturbed Dirac operator H coincides with $\sigma(H_0)$ and the perturbation Q can provoke a non-trivial point spectrum. Our problem is to study the point spectrum of the perturbed Dirac operator H . This problem has been studied by many researchers in connection with various problems (note that the most of the results were concerned with the case $n = 3$ and $m = 4$). A good deal of background material on the development and perspectives of the problem can be found in [1], [2], [3], [5], [7], [10], [12], [13], [14]. Apart from the already mentioned works, we refer to the [15] and the references given therein for a partial list.

In this paper, we give conditions on $Q(x)$ under which the point spectrum of H (if any) has ± 1 as the only possible accumulation points. Specifically, we assume that $Q(x)$ satisfies the following assumption.

(A) $Q(x) = [q_{jk}(x)]$, $x \in \mathbb{R}^n$, is an $m \times m$ Hermitian matrix-valued function the entries of which are elements from the space $L_\infty(\mathbb{R}^n)$ and

$$\lim_{|x| \rightarrow \infty} |x| q_{jk}(x) = 0 \quad (j, k = 1, \dots, m).$$

The main results are obtained by applying the abstract results from [6] (see also its refinement results made in [9]). Below, we cite the corresponding result.

Let \mathcal{H} be a Hilbert space. Denote by $\mathbb{B}(\mathcal{H})$ the space of all bounded operators on \mathcal{H} and by $\mathbb{B}_\infty(\mathcal{H})$ the subspace of $\mathbb{B}(\mathcal{H})$ consisting of all compact operators in \mathcal{H} . The domain and the range of an operator A are denoted by $\text{Dom}(A)$ and $\text{Ran}(A)$, respectively.

Theorem 1.1. [9] *Let A and B be symmetric operators in a space \mathcal{H} and let the operator A has no eigenvalues on a closed interval Λ of the real axis. Suppose that there exists an operator-valued function $T(\lambda)$ defined on the interval Λ having the properties that*

- (i) $T(\lambda) \in \mathbb{B}_\infty(\mathcal{H})$ ($\lambda \in \Lambda$),
- (ii) $T(\lambda)$ is continuous on Λ in the uniform norm topology, and
- (iii) for each $\lambda \in \Lambda$ and for each $u \in \text{Dom}(B)$ such that $Bu \in \text{Ran}(A - \lambda I)$ there holds the following inequality

$$\| (A - \lambda I)^{-1} Bu \| \leq \| T(\lambda)u \| . \quad (1.5)$$

Then the point spectrum of the perturbed operator $A + B$ on the interval Λ consists only of finite number of eigenvalues of finity multiplicity.

Remark 1.2. *The assertion of Theorem 1.1 remains true if in place of (1.5) it is required the following one*

$$\| (A - \lambda I)^{-1} Bu \| \leq \sum_{k=1}^N \| T_k(\lambda)u \| , \quad (1.6)$$

where the operator-valued functions $T_k(\lambda)$ ($k = 1, \dots, N$) satisfy the conditions (i) and (ii).

As we already mentioned we will apply Theorem 1.1 to the study of the problem of the discreteness of the set of eigenvalues of the perturbed Dirac operator H . The main results are presented in the next section.

2. Main results

Let H be the Dirac operator defined by (1.1) in which the matrix-valued function satisfies the assumption (A). The unperturbed Dirac operator H_0 represents a matrix differential operator (of the dimension $m \times m$) of order 1. The symbol of the operator H_0 is a matrix-valued function which we denote by $h_0(\xi)$, $\xi \in \mathbb{R}^n$. Note that by applying the Fourier transformation to the elements of the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ the operator H_0 is transformed (in the momentum space) into a multiplication operator by the matrix $h_0(\xi)$. The Fourier transformation is defined by the formula

$$\hat{u}(\xi) = (Fu)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int u(x) e^{i\langle x, \xi \rangle} dx \quad (u \in L_2(\mathbb{R}^n))$$

in which $\langle x, \xi \rangle$ designates the scalar product of the elements $x, \xi \in \mathbb{R}^n$ (here and in what follows $\int : \int_{\mathbb{R}^n}$). The corresponding norm in \mathbb{R}^n (or \mathbb{C}^m) will be denoted as usually by $|\cdot|$. The operator norm of $m \times m$ matrices corresponding to the norm $|\cdot|$ in \mathbb{C}^m will be denoted by $|\cdot|$, as well.

Our main result is the following

Theorem 2.1. *Let H be the perturbed Dirac operator defined by (1.1) for which the assumption (A) is satisfied. Then the point spectrum of the operator H has only ± 1 as accumulation points. Each eigenvalue can be only of a finite multiplicity.*

Proof. That the spectrum in the spectral gap $(-1, 1)$ is only discrete without any accumulation points in the interior of this interval follows at once due to Weyl type theorems. Let Λ be an closed interval contained in the set $(-\infty, -1) \cup (1, +\infty)$ and let λ be an arbitrary point belonging to Λ . It will be shown that under assumed conditions the operators H_0 and H verify all of hypotheses of Theorem 1.1. To this end, we estimate the norm of the element $(H_0 - \lambda I)^{-1}Qu$ for each $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ such that $Qu \in \text{Ran}(H_0 - \lambda I)$. Let \widehat{Qu} be the Fourier transform of Qu , and denote

$$\widehat{v}(\xi) := (h_0(\xi) + \lambda)\widehat{Qu}(\xi), \quad \xi \in \mathbb{R}^n.$$

According to (1.4), we may write

$$\begin{aligned} \|(H_0 - \lambda I)^{-1}Qu\|^2 &= \int |(h_0(\xi) - \lambda)^{-1}\widehat{Qu}(\xi)|^2 d\xi \\ &= \int (|\xi|^2 - r(\lambda)^2)^{-1}|\widehat{v}(\xi)|^2 d\xi, \end{aligned} \quad (2.1)$$

where $r(\lambda) := \sqrt{\lambda^2 - 1}$.

Next, we let

$$\Omega(\Lambda) = \cup_{\lambda \in \Lambda} \{\xi \in \mathbb{R}^n : |\xi| = r(\lambda)\}$$

and we choose a sphere U of radius R with center of the origin such that $U \supset \Omega(\Lambda)$ and let $V = \mathbb{R}^n \setminus U$. Then passing to spherical coordinates $\xi = |\xi|\omega$, $\rho = |\xi|$ (we write ds_ω for the area element of hypersurface S_{n-1} of the unit sphere S in \mathbb{R}^n), and denoting

$$\widehat{f}(\rho, \omega) = \frac{\rho^{\frac{n-1}{2}} \widehat{v}(\rho\omega)}{\rho + r(\lambda)} \quad (0 \leq \rho < \infty, \omega \in S_{n-1}),$$

we have

$$\begin{aligned} \int_U |(h_0(\xi) - \lambda)^{-1}\widehat{Qu}(\xi)|^2 d\xi &= \int_{S_{n-1}} \int_0^R \rho^{n-1} |(\rho^2 - r(\lambda)^2)^{-1} \widehat{v}(\rho\omega)|^2 d\rho dS_\omega \\ &= \int_{S_{n-1}} \int_0^R \left| \frac{\widehat{f}(\rho, \omega)}{\rho - r(\lambda)} \right|^2 d\rho dS_\omega. \end{aligned}$$

Since $Qu \in \text{Ran}(H_0 - \lambda I)$, it follows that $\widehat{f}(\rho, \omega)$ vanishes at $\rho = r(\lambda)$, and we can continue

$$\begin{aligned} &\left[\int_{S_{n-1}} \int_0^R \left| \frac{\widehat{f}(\rho, \omega)}{\rho - r(\lambda)} \right|^2 d\rho dS_\omega \right]^{\frac{1}{2}} \\ &= \left[\int_{S_{n-1}} \int_0^R \left| \int_0^1 \frac{\partial \widehat{f}}{\partial \rho}(t(\rho - r(\lambda)) + r(\lambda), \omega) dt \right|^2 d\rho dS_\omega \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^1 \left[\int_{S_{n-1}} \int_0^R \left| \frac{\partial \hat{f}}{\partial \rho} (t(\rho - r(\lambda)) + r(\lambda), \omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}} dt \\
 &\leq 2 \left[\int_{S_{n-1}} \int_0^R \left| \frac{\partial \hat{f}}{\partial \rho} (\rho, \omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}} \\
 &\leq \left[\int_{S_{n-1}} \int_0^R \left| \rho^{\frac{n-3}{2}} ((n-3)\rho + (n-1)r(\lambda))(\rho + r(\lambda))^{-2} \hat{v}(\rho\omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}} \\
 &\quad + 2 \left[\int_{S_{n-1}} \int_0^R \left| \rho^{\frac{n-1}{2}} (\rho + r(\lambda))^{-1} \frac{\partial}{\partial \rho} \hat{v}(\rho\omega) \right|^2 d\rho ds_\omega \right]^{\frac{1}{2}}.
 \end{aligned}$$

Taking into account that $|\frac{\partial}{\partial \rho} \hat{v}(\rho\omega)| \leq |\nabla \hat{v}|$, we get

$$\begin{aligned}
 &\left[\int_U \left| (h_0(\xi) - \lambda)^{-1} \widehat{Q}u(\xi) \right|^2 d\xi \right]^2 \\
 &\leq 2r(\lambda) \left[\int_U \left| \frac{(n-3)|\xi| + (n-1)r(\lambda)}{|\xi|(|\xi| + r(\lambda))^2} \hat{v}(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} + 2 \left[\int_U \left| \frac{\nabla \hat{v}(\xi)}{|\xi| + r(\lambda)} \right|^2 d\xi \right]^{\frac{1}{2}}.
 \end{aligned}$$

Since the expressions $(n-3)|\xi| + (n-1)r(\lambda)$, $(|\xi| + r(\lambda))^{-1}$ ($\lambda \in \Lambda$; $\xi \in U$) and each element of the matrix-valued function $h_0(\xi) - \lambda$ ($\lambda \in \Lambda$; $\xi \in U$) are bounded on $\Lambda \times U$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}
 &\left[\int_U \left| (h_0(\xi) - \lambda)^{-1} \widehat{Q}u(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} \\
 &\leq c_1 \left[\int_U \left| |\xi|^{-1} \widehat{Q}u(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} + c_2 \left[\int_U |\nabla \widehat{Q}u(\xi)|^2 d\xi \right]^{\frac{1}{2}}.
 \end{aligned}$$

We claim that the integral operators with kernels

$$|\xi|^{-1} Q(x)e^{-i(x,\xi)}, x^l Q(x)e^{-i(x,\xi)} \quad (|l| = 1; x \in \mathbb{R}^n, \xi \in U)$$

are compact operators in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$. The compactness of them can be proved by applying the criteria obtained in [4] (or, also, by applying the lemma from [8], page 45).

In addition, we note that the integral operator K_V with the kernel

$$(h_0(\xi) - \lambda)^{-1} Q(x)e^{-i(x,\xi)} \quad (x \in \mathbb{R}^n; \xi \in V)$$

represents also a compact operator. To see this fact, it suffices to show that

$$\| (I - P_h)K_V \| \rightarrow 0 \text{ as } h \rightarrow \infty, \quad (2.2)$$

where $(P_h u)(x) = u(x)$ for $|x| \leq h$ and $(P_h u)(x) = 0$ for $|x| > h$.

Since each element of the matrix-valued function $(h_0(\xi) - \lambda)^{-1}$ behaves as $|\xi|^{-1}$ at the infinite, it follows the evaluation

$$\| (I - P_h)K_V u \|^2 \leq c \int_{|\xi| > h} |(1 + |\xi|)^{-1} \widehat{Q}u(\xi)|^2 d\xi \leq c(1 + h)^{-2} \| u \|^2,$$

and so (2.2) is realized.

Thus, taking into account (2.1), we obtain an estimate like that from (1.6) (see Remark 1.2) and, therefore Theorem 1.1 can be applied. This completes the proof of Theorem 2.1. \square

As an application of Theorem 2.1 we give a result concerning the particular case of the Hamiltonian of a Dirac particle in an electromagnetic field. The Dirac operator in this case is typically written in the physics literature (see, for instance, [11], [15]) as follows

$$Hu = \sum_{j=1}^3 \alpha_j (D_j - A_j(x))u + \alpha_4 u + q(x)u, u \in W_2^1(\mathbb{R}^3; \mathbb{C}^4), \quad (2.3)$$

where $A(x) = (A_1(x), A_2(x), A_3(x))$ (the vector potential) and $q(x)$ (the scalar potential) are given functions on \mathbb{R}^3 .

Theorem 2.2. *If*

$$\lim_{|x| \rightarrow \infty} |x|A_j(x) = 0 \quad (j = 1, 2, 3), \quad \lim_{|x| \rightarrow \infty} |x|q(x) = 0,$$

then the point spectrum of the Dirac operator defined by (2.1) is discrete having only ± 1 as accumulation points. Each eigenvalue can be only of a finite multiplicity.

References

- [1] A. Berthier, V. Georgescu, *On the point spectrum for Dirac operators*, J. Func. Anal. **71** (1987), 309–338.
- [2] M.S. Birman, *On the spectrum of Schrodinger and Dirac operators*, Dokl. Acad. Nauk **129**, no.2 (1959), 239–241.
- [3] M.S. Birman, *On the spectrum of singular boundary-value problems*, Mat. Sb. **55** (1961), no. 2, 125–174.
- [4] M.S. Birman, G.E. Karadzhov, M.Z. Solomyak, *Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs*, Adv. Soviet. Math., **7** (1991), 85–106.
- [5] M.S. Birman, A. Laptev, *Discrete spectrum of the perturbed Dirac operator*, Ark. Mat. **32**(1) (1994), 13–32.
- [6] P.A. Cojuhari, *On the discrete spectrum of a perturbed Wiener-Hopf operator*, Mat. Issled. **54** (1980), 50–55 (Russian).
- [7] P.A. Cojuhari, *On the point spectrum of the Dirac operator*, Dif. Urav., Minsk, Dep. VINITI 24.06.87, N4611-B87, (1987), 1–17 (Russian).
- [8] P.A. Cojuhari, *On the finiteness of the discrete spectrum of some matrix pseudodifferential operators*, Vyssh. Uchebn. Zaved. Mat., **1** (1989), 42–50 (Russian).
- [9] P.A. Cojuhari, *The problem of the finiteness of the point spectrum for self-adjoint operators. Perturbations of Wiener-Hopf operators and applications to Jacobi Matrices*, Operator Theory, Adv. and Appl. **154** (2004), 35–50.
- [10] W.D. Evans, B.J. Harris, *Bounds for the point spectra of separated Dirac operators*, Proc. Roy. Soc. Edinburgh, Sect. A **88** (1980), 1–15.

- [11] A. Fock, *Introduction to Quantum Mechanics*, Nauka, Moscow, 1976 (Russian).
- [12] D.B. Hinton, A.B. Mingarelli, T.T. Read, J.K. Shaw, *On the number of eigenvalues in the spectral gap of a Dirac system*, Proc. Edinburgh Math. Soc. **29**(2) (1986), 367–378.
- [13] M. Klaus, *On the point spectrum of Dirac operators*, Helv. Phys. Acta. **53** (1980), 453–462.
- [14] O.I. Kurbenin, *The discrete spectra of the Dirac and Pauli operators*, Topics in Mathematical Physics, vol. 3, Spectral Theory, (1969), 43–52.
- [15] B. Thaller, *The Dirac Equation*, Springer, Berlin-Heidelberg-New York, 1992.

Petru A. Cojuhari
Department of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewiczza 30
30-059 Cracow, Poland
e-mail: cojuhari@uci.agh.edu.pl