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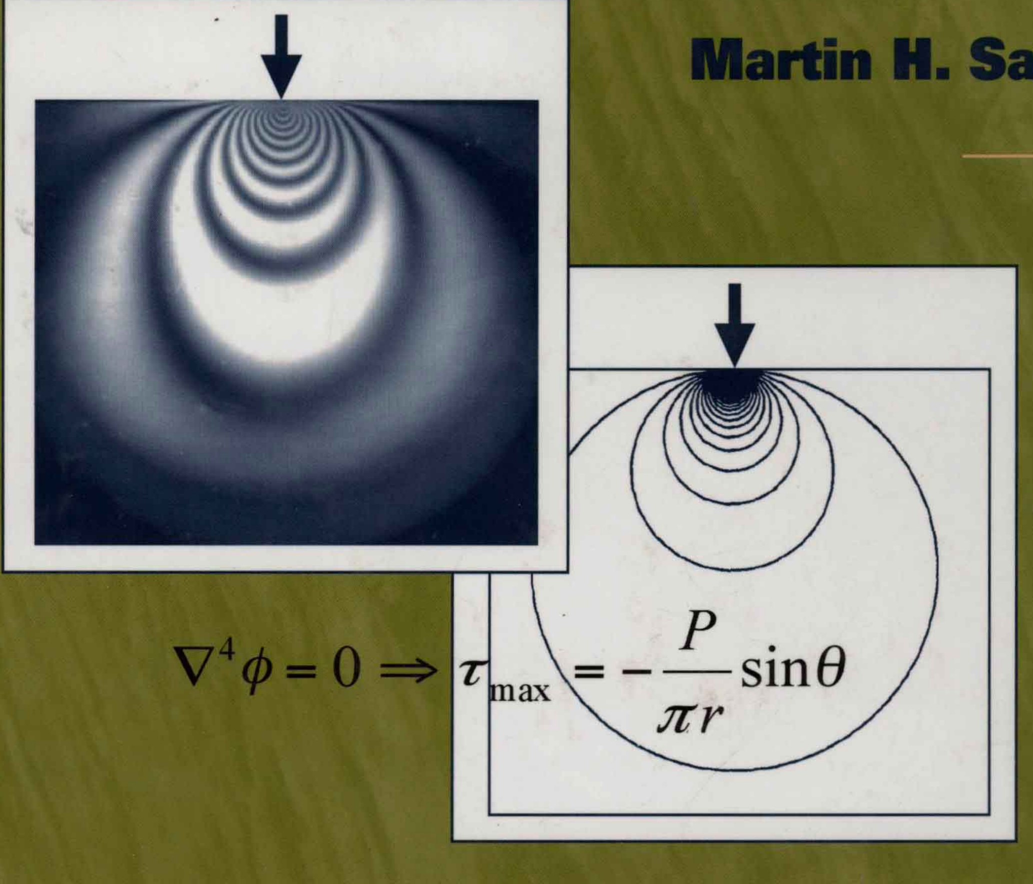
Elasticity

Theory, Applications, and Numerics

Martin H. Sadd



$$\nabla^4 \phi = 0 \Rightarrow \tau_{\max} = -\frac{P}{\pi r}\sin\theta$$

ELASTICITY

Theory, Applications, and Numerics

Second Edition

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Preface

This second edition continues the author's attempt to present linear elasticity with sound, concise theoretical development, numerous and contemporary applications, and enlightening numerics to aid in understanding solutions. In addition to making corrections to typographical errors, several new items have been included. Perhaps the most significant addition is a new chapter on nonhomogeneous elasticity, a topic rarely found in existing elasticity texts. Over the past couple of decades, this field has attracted considerable attention, with engineering interest in the use of functionally graded materials. The new Chapter 14 contains basic theoretical formulations and several application problems that have recently appeared in the literature. A new appendix covering a review of mechanics of materials has also been added, which should help make the text more self-contained by allowing students to review appropriate undergraduate material as needed.

Almost 100 new exercises, spread out over most chapters, have been added to the second edition. These problems should provide instructors with many new options for homework, exams, or material for in-class discussions. Other additions include a new section on curvilinear anisotropic problems and an expanded discussion on interface boundary conditions for composite bodies. The online solutions manual has been updated and corrected and includes solutions to all exercises in this book.

This new edition is again an outgrowth of lecture notes that I have used in teaching a two-course sequence in the theory of elasticity. Part I is designed primarily for the first course, normally taken by beginning graduate students from a variety of engineering disciplines. The purpose of the first course is to introduce students to theory and formulation and to present solutions to some basic problems. In this fashion students see how and why the more fundamental elasticity model of deformation should replace elementary strength of materials analysis. The first course also provides the foundation for more advanced study in related areas of solid mechanics. The more advanced material included in Part II has normally been used for a second course taken by second- and third-year students. However, certain portions of the second part could also be easily integrated into the first course.

What is the justification for my entry of another text in the elasticity field? For many years, I have taught this material at several U.S. engineering schools, related industries, and a government agency. During this time, basic theory has remained much the same; however, changes in problem-solving emphasis, elasticity applications, numerical/computational methods, and engineering education have created the need for new approaches to the subject. I have found that current textbook titles commonly lack a concise and organized presentation of theory, proper format for educational use, significant applications in contemporary areas, and a numerical interface to help develop solutions and understand the results.

The elasticity presentation in this book reflects the words used in the title—*theory, applications,* and *numerics.* Because *theory* provides the fundamental cornerstone of this field, it is important to first provide a sound theoretical development of elasticity with sufficient rigor to give students a good foundation for the development of solutions to a broad class of problems. The theoretical development is carried out in an organized and concise manner in order to not lose the attention of the less mathematically inclined students or the focus of applications. With a primary goal of solving problems of engineering interest, the text offers numerous *applications* in contemporary areas, including anisotropic composite and functionally graded materials, fracture mechanics, micromechanics modeling, thermoelastic problems, and computational finite and boundary element methods. Numerous solved example problems and exercises are included in all chapters.

What is perhaps the most unique aspect of this book is its integrated use of *numerics.* By taking the approach that applications of theory need to be observed through calculation and graphical display, numerics is accomplished through the use of MATLAB®, one of the most popular engineering software packages. This software is used throughout the text for applications such as stress and strain transformation, evaluation and plotting of stress and displacement distributions, finite element calculations, and comparisons between strength of materials and analytical and numerical elasticity solutions. With numerical and graphical evaluations, application problems become more interesting and useful for student learning.

Contents Summary

Part I of the book emphasizes formulation details and elementary applications. Chapter 1 provides a mathematical background for the formulation of elasticity through a review of scalar, vector, and tensor field theory. Cartesian tensor notation is introduced and is used throughout this book's formulation sections. Chapter 2 covers the analysis of strain and displacement within the context of small deformation theory. The concept of strain compatibility is also presented in this chapter. Forces, stresses, and equilibrium are developed in Chapter 3. Linear elastic material behavior leading to the generalized Hooke's law is discussed in Chapter 4, which also briefly discusses nonhomogeneous, anisotropic, and thermoelastic constitutive forms. Later chapters more fully investigate these types of applications.

Chapter 5 collects the previously derived equations and formulates the basic boundary value problems of elasticity theory. Displacement and stress formulations are made and general solution strategies are identified. This is an important chapter for students to put the theory together. Chapter 6 presents strain energy and related principles, including the reciprocal theorem, virtual work, and minimum potential and complementary energy. Two-dimensional formulations of plane strain, plane stress, and antiplane strain are given in Chapter 7. An extensive set of solutions for specific two-dimensional problems is then presented in Chapter 8, and many applications of MATLAB are used to demonstrate the results. Analytical solutions are continued in Chapter 9 for the Saint-Venant extension, torsion, and flexure problems.

The material in Part I provides a logical and orderly basis for a sound one-semester beginning course in elasticity. Selected portions of the text's second part could also be incorporated into such a course.

Part II delves into more advanced topics normally covered in a second elasticity course. The powerful method of complex variables for the plane problem is presented in Chapter 10, and several applications to fracture mechanics are given. Chapter 11 extends the previous isotropic theory into the behavior of anisotropic solids with emphasis on composite materials. This is an important application, and again, examples related to fracture mechanics are provided. Curvilinear anisotropy has been added in this chapter to explore some basic solutions to problems with this type of material structure.

An introduction to thermoelasticity is developed in Chapter 12, and several specific application problems are discussed, including stress concentration and crack problems. Potential methods, including both displacement potentials and stress functions, are presented in Chapter 13. These methods are used to develop several three-dimensional elasticity solutions.

A new Chapter 14, which covers nonhomogeneous elasticity, has been added. The material in it is unique among standard elasticity texts. After briefly covering theoretical formulations, several two-dimensional solutions are generated along with comparison field plots with the corresponding homogeneous cases. Chapter 15 presents a distinctive collection of elasticity applications to problems involving micromechanics modeling. Included in it are applications for dislocation modeling, singular stress states, solids with distributed cracks, micropolar, distributed voids, and doublet mechanics theories.

Chapter 16 provides a brief introduction to the powerful numerical methods of finite and boundary element techniques. Although only two-dimensional theory is developed, the numerical results in the example problems provide interesting comparisons with previously generated analytical solutions from earlier chapters.

This second edition of *Elasticity* concludes with four appendices that contain a concise summary listing of basic field equations; transformation relations between Cartesian, cylindrical, and spherical coordinate systems; a MATLAB primer; and a new review of the mechanics of materials.

The Subject

Elasticity is an elegant and fascinating subject that deals with determination of the stress, strain, and displacement distribution in an elastic solid under the influence of external forces. Following the usual assumptions of linear, small-deformation theory, the formulation establishes a mathematical model that allows solutions to problems that have applications in many engineering and scientific fields.

- Civil engineering applications include important contributions to stress and deflection analysis of structures, such as rods, beams, plates, and shells. Additional applications lie in geomechanics involving the stresses in materials such as soil, rock, concrete, and asphalt.
- Mechanical engineering uses elasticity in numerous problems in analysis and design of machine elements. Such applications include general stress analysis, contact stresses, thermal stress analysis, fracture mechanics, and fatigue.
- Materials engineering uses elasticity to determine the stress fields in crystalline solids, around dislocations, and in materials with microstructure.
- Applications in aeronautical and aerospace engineering include stress, fracture, and fatigue analysis in aerostructures.

The subject also provides the basis for more advanced work in inelastic material behavior, including plasticity and viscoelasticity, and the study of computational stress analysis employing finite and boundary element methods.

Elasticity theory establishes a mathematical model of the deformation problem, and this requires mathematical knowledge to understand formulation and solution procedures. Governing partial differential field equations are developed using basic principles of continuum mechanics commonly formulated in vector and tensor language. Techniques used to solve these field equations can encompass Fourier methods, variational calculus, integral transforms, complex variables, potential theory, finite differences, finite elements, and so forth. To prepare students for this subject, the text provides reviews of many mathematical topics, and additional references are given for further study. It is important for students to be adequately prepared for the theoretical developments, or else they will not be able to understand necessary formulation details. Of course, with emphasis on applications, we will concentrate on theory that is most useful for problem solution.

The concept of the elastic force-deformation relation was first proposed by Robert Hooke in 1678. However, the major formulation of the mathematical theory of elasticity was not developed until the 19th century. In 1821 Navier presented his investigations on the general equations of equilibrium; he was quickly followed by Cauchy, who studied the basic elasticity equations and developed the notation of stress at a point. A long list of prominent scientists and mathematicians continued development of the theory, including the Bernoullis, Lord Kelvin, Poisson, Lamé, Green, Saint-Venant, Betti, Airy, Kirchhoff, Rayleigh, Love, Timoshenko, Kolosoff, Muskhelishvilli, and others.

During the two decades after World War II, elasticity research produced a large number of analytical solutions to specific problems of engineering interest. The 1970s and 1980s included considerable work on numerical methods using finite and boundary element theory. Also during this period, elasticity applications were directed at anisotropic materials for applications to composites. More recently, elasticity has been used in modeling of materials with internal microstructures or heterogeneity and in inhomogeneous, graded materials.

The rebirth of modern continuum mechanics in the 1960s led to a review of the foundations of elasticity and has established a rational place for the theory within the general framework. Historical details can be found in the texts by Todhunter and Pearson, *History of the Theory of Elasticity*; Love, *A Treatise on the Mathematical Theory of Elasticity*; and Timoshenko, *A History of Strength of Materials*.

Exercises and Web Support

Of special note in regard to this text is the use of exercises and the publisher's website, *www.textbooks.elsevier.com*. Numerous exercises are provided at the end of each chapter for homework assignments to engage students with the subject matter. The exercises also provide an ideal tool for the instructor to present additional application examples during class lectures. Many places in the text make reference to specific exercises that work out details to a particular problem. Exercises marked with an asterisk (*) indicate problems that require numerical and plotting methods using the suggested MATLAB software. Solutions to all exercises are provided online at the publisher's website, thereby providing instructors with considerable help in using this material. In addition, downloadable MATLAB software is available to aid both students and instructors in developing codes for their own particular use to allow easy integration of the numerics.

Feedback

The author is ardently interested in continual improvement of engineering education and definitely welcomes feedback from users of this book. Please feel free to send comments concerning suggested improvements or corrections via surface mail or email (sadd@egr.uri.edu). It is likely that such feedback will be shared with the text's user community via the publisher's website.

Acknowledgments

Many individuals deserve acknowledgment for aiding in the completion of this textbook. First, I would like to recognize the many graduate students who have sat in my elasticity classes. They are a continual source of challenge and inspiration and certainly influenced my efforts to find more effective ways to present this material.

A very special recognition goes to one particular student, Ms. Qingli Dai, who developed most of the original exercise solutions and did considerable proofreading. Several photoelastic pictures have been graciously provided by our Dynamic Photomechanics Laboratory (Professor Arun Shukla, director). Development and production support from several Elsevier staff was greatly appreciated. would also like to acknowledge the support of my institution, the University of Rhode Island, for granting me a sabbatical leave to complete the first edition.

As with the first edition, this book is dedicated to the late Professor Marvin Stippes of the University of Illinois; he was the first to show me the elegance and beauty of the subject. His neatness, clarity, and apparently infinite understanding of elasticity will never be forgotten by his students.

Martin H. Sadd

About the Author

Martin H. Sadd is professor of mechanical engineering and applied mechanics at the University of Rhode Island. He received his Ph.D. in mechanics from the Illinois Institute of Technology in 1971 and then began his academic career at Mississippi State University. In 1979 he joined the faculty at Rhode Island and served as department chair from 1991 to 2000. professor Sadd's teaching background is in the area of solid mechanics with emphasis in elasticity, continuum mechanics, wave propagation, and computational methods. He has taught elasticity at two academic institutions, several industries, and at a government laboratory.

Sadd's research has been in the area of computational modeling of materials under static and dynamic loading conditions using finite, boundary, and discrete element methods. Much of his work has involved micromechanical modeling of geomaterials including granular soil, rock, and concretes. He has authored more than 70 publications and has given numerous presentations at national and international meetings.

Contents

Part I Foundations and Elementary Applications

1 Mathematical Preliminaries

Similar to other field theories such as fluid mechanics, heat conduction, and electromagnetics, the study and application of elasticity theory requires knowledge of several areas of applied mathematics. The theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and this calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is developed as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide a background primarily for the formulation part of our study. Additional review of other mathematical topics related to problem solution technique is provided in later chapters where they are to be applied.

1.1 Scalar, Vector, Matrix, and Tensor Definitions

Elasticity theory is formulated in terms of many different types of variables that are either specified or sought at spatial points in the body under study. Some of these variables are *scalar quantities*, representing a single magnitude at each point in space. Common examples include the material density ρ and temperature T. Other variables of interest are *vector quantities* that are expressible in terms of components in a two- or three-dimensional coordinate system. Examples of vector variables are the displacement and rotation of material points in the elastic continuum. Formulations within the theory also require the need for *matrix variables*, which commonly require more than three components to quantify. Examples of such variables include stress and strain. As shown in subsequent chapters, a three-dimensional formulation requires nine components (only six are independent) to quantify the stress or strain at a point. For this case, the variable is normally expressed in a matrix format with three rows and three columns. To summarize this discussion, in a three-dimensional Cartesian coordinate system, scalar, vector, and matrix variables can thus be written as follows: