A Modern Introduction to

MATHEMATICS

JOHN E. FREUND

A Modern Introduction to

MATHEMATICS

JOHN E. FREUND

Professor of Mathematical Statistics
Virginia Polytechnic Institute

PRENTICE-HALL, INC.

Englewood Cliffs, N.J.

1956

Library of Congress Catalog Card No.: 56-7146

PRINTED IN THE UNITED STATES OF AMERICA

59435

A Modern Introduction to

MATHEMATICS

PRENTICE-HALL MATHEMATICS SERIES

Dr. Albert A. Bennett, Editor

To MICKEY

Preface

THIS BOOK has been written to meet the modern trend in the teaching of mathematics in liberal arts programs. A preliminary draft in mimeographed form was used for a number of years in a course designed to introduce beginners to the fundamentals of mathematics and to develop in them an appreciation for mathematical thinking. Although most readers will probably have had some training in elementary high school algebra and geometry, no particular skill in these subjects is needed as a prerequisite for this book.

Pagewise, one fourth of the text is devoted to the study of numbers, from the natural to the complex numbers. Although this subject is, of course, important in itself, it has been devoted more than the usual space because it affords an excellent opportunity to present the reader with examples and exercises in postulate thinking. If it is desired to speed up this portion of the book, it is possible to omit the discussions of further properties of the various systems of numbers, studying in each case only the postulates and definitions.

In order to pace the rigor of the material covered in the first part of the book, the author has found it desirable to give his students an occasional "time-out" from the more rigorous task at hand. It is for this reason that topics such as number theory, binary, duodecimal, and other scales, and some of the history of mathematical notations are presented informally in Chapters 4, 6, 8, and 10. Any or all of these chapters may be omitted without loss of continuity.

Chapter 2 contains material that is seldom found in a textbook of mathematics. The very brief and informal introduction to the nature of definitions was included with the hope that it will provide some of the necessary interest and motivation for using the postulate approach in the study of mathematics.

Although this book covers many of the traditional topics of algebra, trigonometry, analytic geometry, and a brief introduction to calculus, the emphasis throughout is on basic concepts and ideas. This approach may be more demanding on the reader but it should, at the same time, also be more stimulating and more rewarding than the mere presentation of isolated topics and mere drill in mathematical skills.

In order to maintain the emphasis on the abstract nature of mathematics, some of the later chapters deal with groups, miniature geometries—both introduced at a very elementary level—transfinite numbers, Boolean algebra, and a glimpse at non-Euclidean geometry and topology. Even though the subject of logic is treated in detail near the end of the book in Chapter 23, its first five sections may be studied earlier (perhaps before Chapter 5) if it is desired to use syllogisms as prototypes of mathematical (deductive) proofs. The author has delayed the formal study of logic and Boolean algebra in order to be able to treat these subjects on a slightly more mature level and also to provide a review of the many ideas presented in the early part of the book. Chapters 15, 16, 21, 22, and 23 contain the material mentioned in the beginning of this paragraph and they may be omitted without loss of continuity or taken up in a different order. Another chapter that falls into this category is the chapter on statistics. Chapter 24 is not limited to the elementary formulas of descriptive statistics, but it is aimed to provide the reader with an introduction to the fundamental ideas of statistical inference.

This book has been designed for a 3-hour-per-week two semester (three quarter) course. While teaching with the preliminary mimeographed draft, the author found that if he omitted the three appendices and, perhaps, one or two of the later (optional) chapters, he could easily cover the text in that kind of a course. The extra material on algebra and trigonometry was included primarily to allow for more flexibility in the use of the text. It is felt that with these added topics the reader may pursue further studies in mathematics without having to repeat courses in college algebra and trigonometry. The author also feels that a good student may, perhaps with some outside reading in analytic geometry, proceed directly into a course in calculus.

This text is also suitable for a one-semester course on the foundations of mathematics, and the author would like to suggest that such a course be based on Chapters 2, 3, 5, 7, 9, 11, 14 (Section 14.1 only), 15, 22, and 23. Such a course will cover the development of various systems of numbers, groups, transfinite numbers, and Boolean algebra.

The author would like to thank his many colleagues, students, and other friends whose helpful suggestions, criticisms, and comments contributed greatly to the writing of this book. In particular, the author would like to express his appreciation to Dr. A. A. Bennett, Mr. I. Miller, Dr. C. E. Rhodes, Mr. N. Tiffany, Dr. R. L. Wine, the anonymous colleagues who reviewed the various drafts, and the many professors under whom the author had the pleasure of studying mathematics.

The author is greatly indebted to the Alfred University Research Foundation for their assistance in preparing the mimeographed draft, to Miss Marianne Byrd for her expert typing of the final manuscript, to the editorial staff of Prentice-Hall, Inc., for their courteous cooperation during the production of the book, and to the Monroe Calculating Machine Company and the Allen B. Du Mont Laboratories for providing the photographs which are reproduced as Figures 8.3 and 19.31. Last but not least, the author would like to express his appreciation for the assistance given to him by his wife, to whom this book is affectionately dedicated.

John E. Freund

Contents

Chapter I: Introduction

1.1. *Why study mathematics?*

Many a person must have asked himself the question posed in the heading of this section while struggling through problems in mathematics which he tried to solve in the various stages of his elementary, secondary, and higher education.

Until we start with algebra and forsake the seemingly concrete numbers of arithmetic for abstract symbols it is not difficult to answer this question. We have only to point out what practical advantages there are in being able to count, add, subtract, multiply, and divide. After all, we may some day want to be sure that there are sufficient pieces of pie for a given number of guests, we may some day want to divide two quarts of ice cream evenly among seven friends, or we may some day want to calculate the interest we can expect to get from our bank. Practical examples like these should convince all but a few diehard skeptics that it is desirable to have at least a speaking acquaintance with the subject of arithmetic. Looking at it from this point of view, there are many people who consider the study of mathematics a necessary evil which provides them with a useful but mysterious bag of tricks.

Once we leave the obviously practical phases of mathematics, the problem of presenting an acceptable motivation for its study becomes increasingly difficult. Mathematicians and others have argued for a long time that there is a transfer of knowledge, that the study of mathematics sharpens one's wits and provides a solid foundation for other fields of learning. Whether this is actually the case is debatable, and outstanding scholars can be used to support both sides of this argument. Plato and Aristotle, to mention just two, stressed the general educational value of mathematics, yet Protagoras argued the opposite point of view. More recently there have been committees of educators which issued reports to the effect that there is such a thing as a transfer of knowledge, while others arrived at exactly the opposite conclusion. The current view on this matter can be expressed with the cautious statement that the study of mathematics can sharpen one's ability to understand other fields of learning if it is taught in a special way.

This last qualification is actually the crux of the whole matter. It is easy to understand why people would show a distaste for certain aspects of mathematics if they are asked to perform tasks which they do not really understand, which they cannot justify in their own minds, and which consist of blindly applying mysterious rules laid down by a so-called authority. How can anyone be expected to like and enjoy something he cannot or, at least, does not understand?

The answer to the question about the general value of mathematics seems to be that it has an intellectually broadening influence only if it is taught as a course in which students are exposed to basis concepts and ideas, and not as a course which consists primarily of drills in seemingly arbitrary techniques. All this is, of course, much easier said than done. As a matter of fact, there is a general agreement that students in the lower grades lack the maturity to tackle such an abstract task, and that it is therefore difficult to prevent the early study of mathematics from being much more than drills and memorization. One exception to this is high school geometry, a subject in which we are introduced for the first time to the idea of a rigorous proof, and where we learn to think in terms of theorems and assumptions. Naturally, there is a general value to such a course only if the emphasis is on ideas and not on excessive detail.

If we now consider higher education, we find that many colleges and universities require every freshman to take a year's course in mathematics as part of his general education. Until recent years such courses have often consisted of a sequence of algebra and trigonometry, two subjects which are still required of every student in many institutions. Although there are, of course, exceptions, these courses have all too often contributed