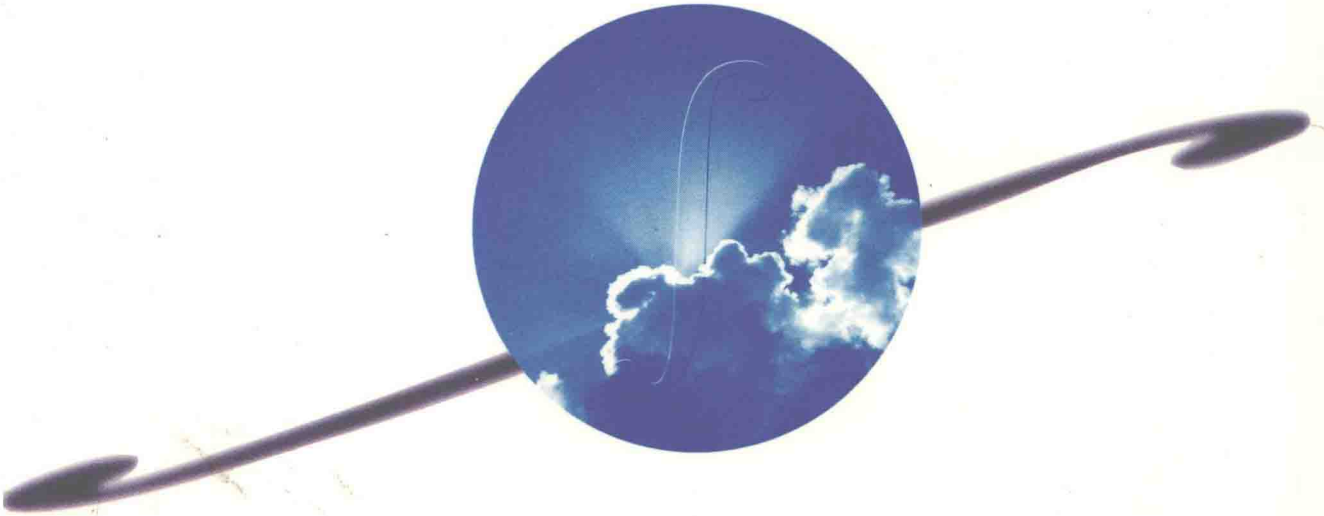
ANDREW F. REX MARTIN JACKSON

INTEGRATED PHYSICS AND CALCULUS



VOLUME I

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Preface

The rules that describe nature seem to be mathematical.
Richard Feynman

Since the time of Galileo, and certainly Newton, physicists have relied upon mathematical models to help them understand natural phenomena. Newton developed calculus so that he could better understand motion and solve the particular problem of the motion of the planets under an inverse-square attractive force. For over 300 years, physicists have used calculus to solve a wide range of problems, including but not limited to those in Newtonian mechanics.

Mathematicians, and teachers of mathematics, have used physical phenomena as examples to help them understand and teach calculus. Calculus is not merely a tool for scientists but a fascinating and rich subject in its own right. Using physics and physical phenomena in the development of calculus helps students develop their intuition and deepen their understanding of the concepts of calculus.

The symbiotic relationship of physics and calculus is what we wish to exploit and enjoy with this book. The two subjects, each an exciting field of study in itself, benefit significantly by being connected. These benefits are often lost, or greatly reduced, when students take separate physics and calculus courses from different departments, with faculty speaking different languages and using different notation, and sequences of topics not synchronized.

Integrated Courses

This textbook contains a full year's course of both physics and calculus, taking the place of two textbooks that present the subjects separately. This unique approach comes just in time for the growing number of courses that emphasize the integration of subjects and connections between disciplines.

In recent years many colleges and universities have found ways to teach physics and calculus in an integrated course or in courses taught separately but somehow coordinated with each other. At some universities the integration or coordination also includes engineering, chemistry, or other subjects, and there are many excellent courses being taught that take advantage of the connections between disciplines. This textbook could be used for any such program that includes physics and calculus, regardless of what other subjects, if any, may be included. Because the physics and calculus topics are integrated in this textbook, it is important that the physics and calculus instructors coordinate the pacing of their classes to take advantage of the integration.

Since 1994, we have taught a successful integrated physics and calculus course at the University of Puget Sound. Our course is team-taught by a physicist and mathematician and meets eight hours per week for a full year. It is intended primarily for freshmen but may include upper-division students who are interested in both subjects. Our freshmen have interests in science, mathematics, engineering, computer science, or some combination thereof, and our course serves all such students. This textbook is based on our experiences teaching that course. In both our course and book we have included the topics that fit together naturally and can be reasonably covered in a full academic year (two semesters or three quarters).

The Integrated Textbook

This textbook assumes that students coming into the course have had some exposure to calculus, either in a high school course or in one semester or quarter of college calculus. Students should thus have had some experience with the concepts of limits and derivatives. We begin the textbook with a review of these topics, which then lead naturally into integral calculus (and simultaneously hang together with the study of kinematics in physics).

We have sought to make connections between physics and calculus wherever such connections could be fruitful, but we have not forced connections in every chapter. For example, the physics and calculus are fully integrated in Chapters 2 through 4, but then Chapter 5 and 7 are mostly physics, while Chapters 6 and 8 are mostly calculus. When there are chapters that are all physics or all calculus, they are close enough in the sequence that the physics and mathematics parts of the course can operate on separate tracks for a while and then come back together for the next integrated part. For example, Chapters 13 and 14 are almost all calculus, and Chapters 15 and 16 almost all physics, but then Chapters 17 and 18 contain both physics and calculus and have significant connections between the two. We leave it up to instructors to decide how best to use the integrated and non-integrated portions of the book.

We take advantage of recent innovations in teaching calculus, specifically by coordinating the use of graphical, numerical, and symbolic points of view. One advantage of this approach is that when calculus is applied to physics, it opens a variety of problems and solutions beyond the strictly symbolic methods used in traditional physics courses. Using graphical, numerical, and symbolic methods enhances students understanding and appreciation for both calculus and physics.

As noted above, we have included only what can be comfortably completed in a full-year course (that meets eight hours per week). Absent are some topics traditionally included in introductory physics, specifically heat and geometrical optics. We have found that these are subjects best studied in the laboratory portion of a course, with calorimeters, thermometers, lamps and thin lenses in hand. For those who find these indispensable in the classroom portion of their course, Addison Wesley Longman offers to custom-print those sections from one of their other calculus-based physics textbooks as a supplement to this book.

Contents Overview

Chapter 1 contains a review of functions, with which the entering student should be familiar. It goes on to introduce vector-output functions. This is a key feature of our text—moving the study of vector-output functions to the beginning of the course, where it can be applied to physics. The first chapter also contains a review of physical dimensions and units, with which most students should be familiar.

In Chapter 2 we review the basics of limit, continuity, and derivative. Although we assume students have seen these topics in their earlier calculus course, they will benefit from seeing them again and also seeing their relation to kinematics (position, velocity, and acceleration). This leads naturally into the development of definite integrals and antiderivatives in Chapter 3 (again with position, velocity, and acceleration in one dimension in mind). In Chapter 4 the discussion is extended to the calculus of vector-output functions along with the kinematics of more than one dimension.

Chapters 5 and 6 make a convenient package. Chapter 5 contains the study of dynamics and Newton's laws of motion. Newton's second law, expressed as a differential equation, leads naturally to the study of differential equations in Chapter 6. In Chapter 7 we consider work and energy. The definition of work, which involves a dot product, is more easily grasped by students who are familiar with vector-output functions. The need to calculate work for a variety of situations motivates our study of both symbolic and numerical antiderivative techniques in Chapter 8. This is a standard topic in most second-semester calculus courses.

We think of Chapters 9–11 as another unit, with Chapters 9 and 10 mostly physics (momentum and rotational motion), and Chapter 11 mostly calculus (sequences and series). These physics and calculus topics are considered essential by most faculty, and we give a full, though not completely integrated, treatment. Chapter 12 is another fully integrated chapter, combining harmonic motion and differential equations. The state space viewpoint is particularly useful in admitting graphical and numerical methods.

Chapters 13 and 14 are devoted mostly to mathematics, and they include much of the material one normally encounters in a third semester of calculus, including an introduction to multivariable functions and multiple integrals. These chapter can be covered simultaneously with Chapters 15 and 16, which are mostly physics. Gravitation (Chapter 15) and electrostatics (Chapter 16) are both governed by an inverse-square distance law. Coulomb's law leads to the introduction of electric fields in Chapter 17, which is integrated with the more general study of vector fields in calculus. Chapter 18 is also fully integrated, with the partial derivative and gradient applicable to electric potential.

The concept of electric potential is then used in the study of capacitors (Chapter 19) and DC circuits (Chapter 21). On the mathematics side Chapters 20 (extrema of multivariable functions) is separate from the physics but can be covered at this time to help prepare for what is to come later, when the subjects are integrated again.

Chapter 22 presents a thorough treatment of line integrals, motivated by their use throughout the course in the context of work and potential energy. Chapter 23 is then fully integrated once again. Gauss's law is presented first. This fundamental law of electromagnetism serves as motivation for the study of surface integrals (for which students have been prepared by the treatment of normal line integrals in Chapter 22).

Chapters 24-26 contain the fundamental laws of electromagnetism (an important part of most second-semester physics courses). In calculus the divergence and curl are introduced in Chapter 25. These vector operators are studied with the geometric interpretation of vector fields in mind, but they also serve to prepare students for Chapter 27, where Maxwell's equations are presented in integral form and then rewritten in differential form with the aid of Stokes's theorem and the divergence theorem from calculus. We stress the interpretation of these theorems as vector calculus versions of the second fundamental theorem of calculus.

We include Chapter 28 as a capstone to the first-year physics experience and a bridge to further study (many institutions follow the first-year course with a course in modern physics). The Bohr model is relatively straightforward (compared say to special relativity) and helps pull together a number of concepts used throughout the year: forces (especially electromagnetic), energy, potential energy, and conservation of energy. While these classical concepts are used, Bohr's radical assumptions give students the understanding that classical physics is not sufficient to explain atomic phenomena, and that a more complete theory is needed. This should help provide the inspiration for further study in physics.

Acknowledgments

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We wish to thank Addison Wesley Longman for their support, and particularly Sami Iwata and Joan Marsh for their editorial and technical assistance. We also recognize Stuart Johnson and Steve Wuerz for their consideration and advice earlier in the project.

Our colleagues Matt Moelter and Bryan Smith, each of whom taught the course while we had sabbaticals, deserve recognition for their hard work and original ideas that contributed to the project. As we have taught the course and developed the textbook, we have benefited significantly from the input of students in the course and other colleagues in our university and elsewhere, who are too numerous to recognize individually.

A special word of thanks goes to Ruark Dreher, who created many of the electronic drawings that appear in the text.

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Last but not least, we acknowledge the support and patience of our families, especially our wives Sharon and Jill. They have helped make this all worthwhile and deserve our deepest gratitude.

Andrew Rex and Martin Jackson
University of Puget Sound
June 1999

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Volume Two

Note: Chapters 13–15 are included in both Volume One and Volume Two.

Chapter 1

Functions and Vectors

Philosophy is written in that great book that always lies before our gaze—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in the language of mathematics, and its symbols are triangles, circles, and other geometrical figures, without the help of which it is impossible to conceive a single word of it, and without which one wanders in vain through a dark labyrinth.

Galileo

The study of calculus begins with the *real numbers*. Real numbers serve as the inputs and outputs of functions in calculus. Real numbers are also used as the model for many physical quantities. A quantity described by a single real number is called a *scalar*. Examples of scalar quantities used in physics include time, mass, and speed. Position in space is an example of a quantity that is described by more than one real number. Such a quantity is called a *vector*. Many other quantities used in physics are vector quantities, including velocity, acceleration, and force. We review some ideas about real numbers and introduce vectors in Section 1.1.

In previous courses, you have studied functions for which each input is a single real number and each output is also a single real number. We review the basic concept of function in Section 1.2.

One of the main new ideas you will encounter in this text is that of a function for which each input and/or output is a vector. Functions with vector inputs or outputs are widely used in modeling physical phenomena. For example, an input of a position function for an object moving in space is a single real number representing time, and the output is a vector representing a position. We introduce these types of functions in Section 1.3.

In Section 1.4, we discuss physical measurement and SI units.

1.1 Real Numbers, Scalars, and Vectors

1.1.1 Real Numbers

Our goal in this section is to develop some intuition for the structure of the set of real numbers, denoted $\mathbb{R}$. Our intuition will come from thinking about $\mathbb{R}$ geometrically in terms of the number line.

The structure of the set of real numbers is surprisingly subtle and beautiful. Developing a full understanding of that structure would require us to define the real numbers starting from some very basic concepts. That effort takes time and would distract from our main path. Instead, we will work with an informal, intuitive view. This approach has the advantage of letting us discuss the powerful concepts and tools of calculus and their applications to physics. The shortcoming is that lacking a complete understanding of the real numbers could lead us to pitfalls in our understanding of the concepts. Fortunately we, the authors, can steer you away from those pitfalls.

In many ways, our approach to the role of real numbers parallels the historical development of calculus. The principal early developers of calculus, Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), had an intuitive notion of the real numbers. They were, nonetheless, able to articulate the fundamental concepts of derivative, definite integral, and their relationship to each other. Some of their contemporaries, most famously George Berkeley (1685–1753), criticized their ideas on the basis of questions raised by the lack of rigor. Later mathematicians, among them Leonhard Euler (1707–1783) and Joseph-Louis Lagrange (1736–1813), successfully applied the ideas of calculus in mathematics and physics, but they too encountered subtle questions that arose from the same lack of rigor. These questions lead Augustin Cauchy (1789–1857) and others to develop a clearer understanding of the function concept and the limit process that lie at the heart of calculus. The work of these mathematicians was hindered by the lack of a complete understanding of the real numbers. After Julius Dedekind (1831–1916) and Georg Cantor (1845–1918) laid out rigorous methods of defining the real numbers, Karl Weierstrass (1815–1897) was able to give a full definition of the limit concept that is free of ambiguity.

To understand the real numbers geometrically, we use a number line. That is, we envision a straight line that extends in both directions without end. We choose some point as our origin and label it with the real number zero. We then choose some distance to be one unit and label the point that is one unit to the right of zero as the number 1, the point two units to the right as 2, and so on. We label the point one unit to the left of zero with -1, two units to the left with -2, and so on. This gives us a correspondence between the **integers** $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$ and certain points on the number line, as shown in Figure 1.1(a). We can divide the interval between any two integer points into equal-size subdivisions and label the points at the ends of those subdivisions with **rational numbers**, that is, numbers that are ratios of two integers. For example, we could divide the interval between 2 and 3 into five equal-size subdivisions and look at the right end of the fourth subdivision as shown in Figure 1.1(b). We would label this point as $14/5$.

(a) A generic number line. (b) The number $14/5$ plotted on a number line.

Figure 1.1. A number line used to view real numbers geometrically.

The subtlety of the real numbers becomes apparent if we ask the following question: Can every point on the number line be labeled with a rational number? The answer is no. For example, we can prove that the number that multiplied by itself gives 2 is not a rational number (i.e., the square roots of 2 are not rational numbers). The **irrational numbers** fill in the "gaps" on the number line. One way to characterize the irrational numbers is in terms of decimal representations. A rational number has a decimal representation with a finite number of digits or a repeating pattern with a finite number of digits. An irrational number has a decimal representation that does not end and does not contain a repeating pattern of finite size. This characteristic makes it quite difficult to get a good feel for irrational numbers. Most of our experience with irrational numbers comes at the level of dealing with a symbol, such as $\sqrt{2}$ or π, that represents an irrational number. We often think of the corresponding number in terms of a rational number that approximates it. For example, we often think of $\sqrt{2}$ as approximately 1.41 and π as approximately 3.14 or 22/7. Because the collection of named irrational numbers most of us know is small, it is normal to think that there are not very many irrationals. One of the great surprises revealed by a careful study of the real numbers is that there are, in a well-defined sense, many more irrational numbers than rational numbers. One way to think about this is to imagine putting your finger (or the point of an "ideal" needle) on precisely one point of the number line at random. Chances are greater that this point would correspond to an irrational number rather than a rational number.

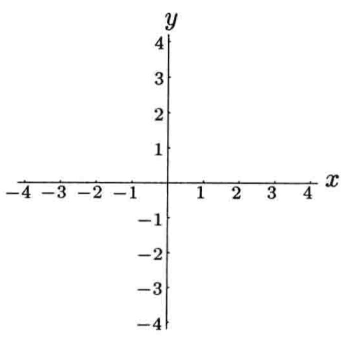
The rational numbers and the irrational numbers together make up the set of real numbers. The set of real numbers is **complete**, which we will take to mean that every point on the number line can be labeled with a real number.

For the set of real numbers, we have the basic operations of addition and multiplication. The number 0 has a special role as the **identity for addition**, and the number 1 has a special role as the **identity for multiplication**. Subtraction and division can be viewed as inverses of the basic operations. The algebraic structure of the real numbers is summarized in familiar rules for commutivity, associativity, and distributivity. These are given in Table 1.1.

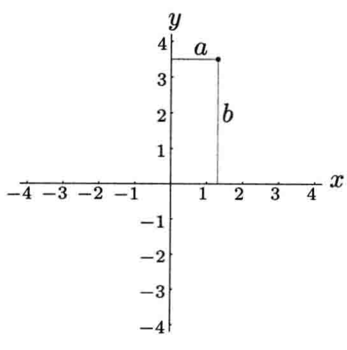
Table 1.1. Rules of algebra for $\mathbb{R}$ (x, y, and z represent real numbers)

Name	Rule
commutative law of addition	$x + y = y + x$
associative law of addition	$(x + y) + z = x + (y + z)$
commutative law of multiplication	$xy = yx$
associative law of multiplication	$(xy)(z) = x(yz)$
distributive law	$x(y + z) = xy + xz$

With the real numbers in hand, we can extend our view to look at **ordered pairs** of real numbers. An ordered pair is just what it says: a pair of numbers (a, b) given in a particular order so that, for example, the pair $(3.1, 4.5)$ is distinct from the pair $(4.5, 3.1)$. We can visualize ordered pairs of real numbers with a natural extension of the number line idea. Now we think of two number lines oriented perpendicular to each other with both origins at the same location as shown in Figure 1.2(a). This is, of course, familiar as a Cartesian coordinate system for the plane. The correspondence between an ordered pair (a, b) and a point on



(a) A standard Cartesian coordinate system

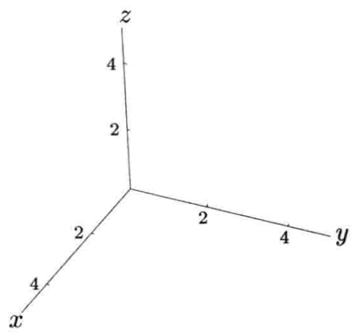


(b) The Cartesian coordinates (a, b) of a point in $\mathbb{R}^2$

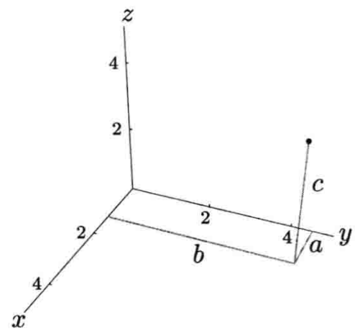
Figure 1.2. Ordered pairs of real numbers can be interpreted geometrically as Cartesian coordinates of points in a plane.

a plane is given in the standard way [as shown in Figure 1.2(b)] by the intersection of the vertical line drawn through the point a on the horizontal axis with the horizontal line drawn through the point b on the vertical axis. The set of all ordered pairs of real numbers is geometrically realized as a plane. We denote the set of all ordered pairs by the symbol $\mathbb{R}^2$ (read "r-two," not "r-squared"). Note that the superscript 2 here does not refer to squaring (at least in the ordinary sense of multiplication).

In a similar fashion, we can consider **ordered triples** of real numbers. An ordered triple (a, b, c) can be visualized geometrically by constructing a Cartesian coordinate system with three mutually perpendicular number lines having origins at the same location as shown in Figure 1.3(a). In Figure 1.3(b), we show an example of the Cartesian coordinates of the point corresponding to an ordered triple. The set of all ordered triples is geometrically realized as (three-dimensional) space. Extending our prior notation, we denote the set of all ordered triples by $\mathbb{R}^3$ (read "r-three").



(a) A standard Cartesian coordinate system



(b) The Cartesian coordinates (a, b, c) of a point in $\mathbb{R}^3$

Figure 1.3. Ordered triples of real numbers can be interpreted geometrically as Cartesian coordinates of points in space.

1.1.2 Scalars and Vectors

The physical quantities we encounter in this text can be classified using two categories: **scalars** and **vectors**. For example compare the natures of temperature and position. To specify a temperature, we can say "it is 21 degrees Celsius outside today." This report contains a single number (with physical units included). To specify a position, we can say "my house is 3 kilometers north and 2 kilometers west of the school." This description contains two numbers, the distance north and the distance west. The need for more than one number distinguishes quantities such as position from quantities such as temperature. **A scalar is a quantity specified with a single number; a vector is a quantity specified with more than one number.**

We can develop an understanding of vectors in two ways: symbolic/numeric and geometric. Seen from the symbolic/numeric view, a vector is essentially an ordered pair or triple of the type we introduced above. From the geometric view, a vector is an arrow of given length and direction. We first develop the geometric view and then relate it to the symbolic/numeric view. In the end, we will have an understanding of vectors that represent physical quantities *and* a geometric view of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

We start with vectors that lie on a line. Imagine a streetcar that runs back and forth along a straight section of track 6 kilometers long. We can model the track with a number line whose origin represents one end of the track, as shown in Figure 1.4(a). Consider a situation in which the streetcar is initially stopped at the 2-kilometer mark. We represent this first position of the streetcar with respect to the origin with an arrow as

(a) A model for the steetcar track

(b) A vector representing the first position of the streetcar

Figure 1.4. Position of a streetcar modeled as a vector quantity.

shown in Figure 1.4(b). The notation that lets us name this vector with a symbol and refers to its numerical value uses an arrow over the symbol. For this example, we denote the position vector as $\vec{a}$. The arrow reminds us that we are thinking of this quantity as vector rather than as a scalar. (In many texts, boldface is used to indicate vector quantities. Using that convention, we would label the vector **a**.)

The numerical value of a vector is given inside a pair of angle brackets $\langle \ \rangle$. For this example, the position vector has a magnitude of 2 kilometers and is pointing to the right, which we denote as the positive direction. Thus we write

$$\vec{a} = \langle +2 \text{ km} \rangle.$$

As is common, we will drop the positive sign and only display negative signs explicitly. It is also common to move the unit names outside of the brackets, so we have

$$\vec{a} = \langle 2 \rangle \text{ km}.$$

As this point, it is reasonable to ask "What is the point of putting these angle brackets around the 2?" Looking at how we extend this notation to vectors that lie in a plane or in space makes this become more sensible.

Imagine a ship under sail from one port to another port. An island blocks the direct path as shown in Figure 1.5. Note that the figure includes a Cartesian coordinate system with a chosen origin. Suppose the ship has reached a particular position. Let's draw an arrow from the origin to represent this position as a vector. This arrow has a specific **magnitude** and points in a specific **direction**. We can give the vector a symbolic name, say $\vec{p}$. This vector extends 5 kilometers in the x-direction and 3 kilometers in the y-direction. We denote this using the angle brackets as

$$\vec{p} = \langle 5, 3 \rangle \text{ km}.$$

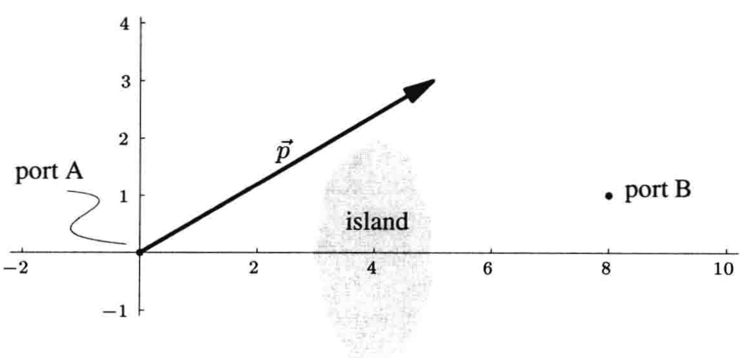


Figure 1.5. The vector $\vec{p}$ represents the initial position of the ship.

In this case of a vector in the plane, the angle brackets have a purpose as the ends of a package. In both cases, the angle brackets also remind us to think about the quantity at hand as a vector so that geometrically we picture arrows rather than points.

It is useful to have some terminology at this stage. When we write a vector in the plane as $\vec{a} = \langle a_1, a_2 \rangle$, we refer to the scalars a_1 and a_2 as the **components** of $\vec{a}$. It is sometimes convenient to use the axis names of the coordinate system as the subscripts for the components. For example, if we label the axes as the x-axis and the y-axis, we write $\vec{a} = \langle a_x, a_y \rangle$. We refer to a_x as the x-component of $\vec{a}$ and a_y as the y-component of $\vec{a}$.

1.1.3 Vector Algebra

Let's return to the streetcar example. Suppose the streetcar has moved to a second position at the 6-kilometer mark, as shown in Figure 1.6. We use another arrow to represent this second position and label it $\vec{c}$. This vector has the numerical value

$$\vec{c} = \langle 6 \rangle \text{ km}.$$

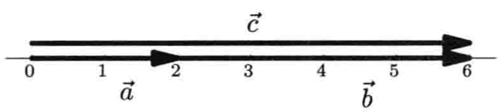


Figure 1.6. Vectors representing positions and displacement of the streetcar.

Now we can describe the **displacement** of the streetcar in its movement from the first position to the second position. Displacement is simply the distance and direction that the streetcar moved in going from the first to the second position. The displacement is also a vector quantity; we denote it as $\vec{b}$. Because the tail of $\vec{b}$ is at the 2 km mark and the head of $\vec{b}$ is at the 6-km mark, we see that $\vec{b}$ has a length of 4 km and points in the positive direction. Thus

$$\vec{b} = \langle 4 \rangle \text{ km}.$$

Note that the tail of a vector need not be placed at the origin. The information content of a vector is given by the length and direction, not by its placement with respect to a coordinate system.

Think about the relationship between the two positions and the displacement. From the physical meaning of these quantities, we could say the following: The second position should be the result of *adding* the displacement to the first position. The point of this observation is to lend some credence to *defining* addition of vectors in the following manner.

Definition 1.1. The **sum** $\vec{a} + \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is the vector drawn from the tail of the arrow $\vec{a}$ to the head of the arrow $\vec{b}$ when $\vec{b}$ is drawn with its tail at the head of $\vec{a}$.

With this definition for adding vectors, we can write a relation between the positions of the streetcar and the displacement as

$$\vec{c} = \vec{a} + \vec{b}.$$

On the numerical level, this is expressed as

$$\langle 6 \rangle \text{ km} = \langle 2 \rangle \text{ km} + \langle 4 \rangle \text{ km}.$$