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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LÉVY NOISE

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An Evolution Equation Approach

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Preface

This book is an introduction to the theory of stochastic evolution equations with Lévy noise. The theory extends several results known for stochastic partial differential equations (SPDEs) driven by Wiener processes. We develop a general framework and discuss several classes of examples both with general Lévy noise and with Wiener noise. Our approach is functional analytic and, as in Da Prato and Zabczyk (1992a), SPDEs are treated as ordinary differential equations in infinite-dimensional spaces with irregular coefficients. In many respects the Lévy noise theory is similar to that for Wiener noise, especially when the driving Lévy process is a square integrable martingale. The general case reduces to this, owing to the Lévy–Khinchin decomposition. The functional analytic approach also allows us to treat equations with a so-called cylindrical Lévy noise and implies, almost automatically, that solutions to equations with Lévy noise are Markovian. In some important cases, however, a càdlàg version of the solution does not exist.

An important role in our approach is played by a generalization of the concept of the reproducing kernel Hilbert space to non-Gaussian random variables, and its independence of the space in which the random variable takes values. In some cases it proves useful to treat Poissonian random measures, with respect to which many SPDEs have been studied, as Lévy processes properly defined in appropriate state spaces.

The majority of the results appear here for the first time in book form, and the book presents several completely new results not published previously, in particular, for equations driven by homogeneous noise and for dissipative systems.

Several monographs have been devoted to stochastic ordinary differential equations driven by discontinuous noise: see Métivier (1982), Protter (2005), Applebaum (2005) and Cont and Tankov (2004), the last two of which are devoted entirely to the case of Lévy noise.

To the best of our knowledge the only monograph devoted to SPDEs with general noise is Kallianpur and Xiong (1995), which covers mainly linear equations.

The papers by Chojnowska-Michalik (1987) on Ornstein–Uhlenbeck processes and by Kallianpur and Pérez-Abreu (1988) were the first to discuss SPDEs with Lévy noise. Then, after a period of 10 years, articles on the subject started to appear again; see e.g. Albeverio, Wu and Zhang (1998), Applebaum and Wu (2000), Bo and Wang (2006), Fournier (2000, 2001), Fuhrman and Röckner (2000), Hausenblas (2005), Mueller (1998), Mytnik (2002), Knoche (2004), Saint Loubert Bié (1998), Stolze (2005) and Peszat and Zabczyk (2006).

Infinite-dimensional calculus, but not SPDEs, with Lévy noise in a disguised form appeared in the late 1990s, in papers on mathematical finance devoted to the bond market; see Björk *et al.* (1997), Björk, Kabanov and Runggaldier (1997) and Eberlein and Raible (1999). This list is certainly not exhaustive.

The book starts with an introductory chapter outlining the interplay between stochastic dynamical systems, Markov processes and stochastic equations with Lévy noise. It turns out that all discrete-time stochastic dynamical systems on arbitrary linear Polish spaces can be represented as solutions to stochastic difference equations, in which the noise, of random-walk type, enters the equation linearly. An analogous situation occurs for continuous-time stochastic dynamical systems on $\mathbb{R}^d$. Here the noise is a Lévy process and the stochastic difference equation is replaced by a stochastic equation of Itô type. To some extent this is also true in infinite dimensions. That is why stochastic evolution equations with Lévy noise are of particular interest.

Chapters 2 and 3 are devoted respectively to analytic and probabilistic preliminaries. Basic definitions related to differential operators and function spaces are recalled, together with fundamental concepts from the theory of stochastic processes in finite- and infinite-dimensional spaces.

Lévy processes in infinite-dimensional spaces are studied in Chapter 4. The chapter starts with explicit constructions of Wiener and Poisson processes. Then it deals with the Lévy–Khinchin decomposition and the Lévy–Khinchin formula. Integrability properties are also studied.

In Chapter 5 transition semigroups of Lévy processes are considered and, in particular, their generators.

The important concept of a Poisson random measure is discussed in Chapter 6. An application to the construction of Lévy processes is given as well. Some moment estimates are derived.

In Chapter 7 we introduce the concept of the reproducing kernel Hilbert space (RKHS) of a square integrable Lévy process. Then we study so-called cylindrical processes and calculate their reproducing kernels. It is also shown that Poisson random measures can be treated as Lévy processes with values in sufficiently large Hilbert spaces. This identification is behind the majority of the results in this book. It is also shown that cylindrical processes are distributional derivatives of Lévy sheets.

Chapter 8 concerns stochastic integration, first with respect to square integrable Hilbert-space-valued martingales and, as an application, with respect to general Lévy processes. The construction of the operator angle bracket is explained and a class of integrands is characterized. The final sections are devoted to integration with respect to a Poisson measure and integration in L^p-spaces.

Part II of the book deals with the existence of solutions and their regularity. Chapter 9 starts with a semigroup treatment of the Cauchy problem for deterministic evolution equations. Next, weak solutions and mild solutions to stochastic equations are introduced and their equivalence is established. The existence of weak solutions to linear equations is proven as well. If the noise evolves in the state space then càdlàg regularity of the solution is proved, using the Kotelenez maximal inequality and, in parallel, a dilation theorem, as in Hausenblas and Seidler (2006). We provide an example which shows that the solutions are not càdlàg in general. Finally, the existence of weak solutions is established and their Markov property is proved.

In Chapter 10 we show that in some cases the Lipschitz assumption can be relaxed.

Chapter 11 is devoted to the so-called factorization method, introduced in Da Prato, Kwapień and Zabczyk (1987). The method allows us, in particular, to prove the continuity of the stochastic convolution of an arbitrary semigroup with a Wiener process.

In Chapters 12 and 13, the general theory of the previous chapters is applied to stochastic parabolic problems and to stochastic wave, delay and transport equations. Lévy noise is then treated as a Lévy process in extended, Sobolev-type, spaces. Parabolic equations of a similar type were dealt with in, for example, Albeverio, Wu and Zhang (1998), Saint Loubert Bié (1998) and Applebaum and Wu (2000). They are discussed here in a unified way and for general partial differential operators. Sharp regularity results for the Wiener noise, using the factorization method, are obtained as well. Stochastic wave equations driven by Lévy processes have not been studied previously in the literature.

In Chapter 14 we develop a theory of stochastic equations with spatially homogenous Lévy noise of both parabolic and hyperbolic type on $\mathbb{R}^d$. Results known already for Wiener noise are extended to this more involved case.

The final chapter of the second part of the book, Chapter 15, is devoted to equations in which the noise enters through the boundary.

Part III is devoted to selected applications. Our aim is to show the applicability of the theory to specific models of physical and economic character. In particular, models in statistical mechanics, fluid dynamics and finance are studied in greater detail. In Chapter 16 we give a self-contained treatment of invariant measures for dissipative systems with Lévy noise and in Chapter 17 we consider lattice systems. The Burgers equation is studied in Chapter 18, and, after a brief discussion of a model for environmental pollution in Chapter 19, in Chapter 20 we present some applications of stochastic infinite-dimensional analysis to mathematical models of the bond market.

In Appendix A we give proofs of some results on linear operators often used in the text. In Appendix B we gather basic results on the theory of C_0-semigroups and provide important results on specific semigroups used in the book. Special attention is paid to semigroups with non-local generators. This allows us to extend the results proved for equations with differential operators. However, owing to space limitations, stochastic equations with non-local linear parts are not discussed in this book. In Appendix C the existence of càdlàg versions of Markov processes is proved. This leads to a simple proof of the existence of a càdlàg version of an arbitrary Lévy process. In Appendix D we recall the Itô formulae for semimartingales.

A list of symbols is given before the Index.

This book grew out of lectures and papers presented by the authors. We have used some material from Peszat and Zabczyk (2004) as well as many unpublished notes.

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Contents

Part I

Foundations

1

Why equations with Lévy noise?

The book is devoted to stochastic evolution equations with Lévy noise. Such equations are important because, roughly speaking, stochastic dynamical systems, or equivalently Markov processes, can be represented as solutions to such equations. In this introductory chapter, it is shown how that is the case. To motivate better the construction of the associated stochastic equations, the chapter starts with discrete-time systems.

1.1 Discrete-time dynamical systems

A deterministic discrete-time dynamical system consists of a set E, usually equipped with a σ-field $\mathcal{E}$ of subsets of itself, and a mapping F, usually measurable, acting from E into E. If the position of the system at time $t = 0, 1, \ldots,$ is denoted by $X(t)$ then by definition $X(t + 1) = F(X(t))$, $t = 0, 1, \ldots$ The sequences $(X(t),\ t = 0, 1, \ldots)$ are the so-called *trajectories* or *paths* of the dynamical system, and their asymptotic properties are of prime interest in the theory. The set E is called the *state space* and the transformation F determines the dynamics of the system.

If the present state x determines only the probability $P(x, \Gamma)$ that at the next moment the system will be in the set Γ then one says that the system is stochastic. Thus a *stochastic dynamical system* consists of the state space E, a σ-field $\mathcal{E}$ and a function $P = P(x, \Gamma)$, $x \in E$, $\Gamma \in \mathcal{E}$, such that, for each $\Gamma \in \mathcal{E}$, $P(\cdot, \Gamma)$ is a measurable function and, for each $x \in E$, $P(x, \cdot)$ is a probability measure. We call P the *transition function* or *transition probability*. A deterministic system is a particular case of a stochastic system with $P(x, \cdot) = \delta_{F(x)}$, where δ_r denotes the Dirac measure at r. We define, by induction, the *probability of visiting* sets

$\Gamma_1, \ldots, \Gamma_k$ at times $1, \ldots, k$, starting from x by

$$P(x, \Gamma_1, \ldots, \Gamma_k) = \int_{\Gamma_1} P(x, \mathrm{d}x_1) P(x_1, \Gamma_2, \ldots, \Gamma_k).$$

The stochastic analogue of the trajectory of a deterministic dynamical system is called a Markov chain.

Definition 1.1 A *Markov chain* X with transition probability P starting from $x \in E$ is a sequence $(X(t),\ t = 0, 1, \ldots)$ of E-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

(i) $X(0) = x, \qquad \mathbb{P}$ -a.s.,
(ii) $\mathbb{P}\big(X(j) \in \Gamma_j,\ j = 1, \ldots, k\big) = P(x, \Gamma_1, \ldots, \Gamma_k), \quad \forall \Gamma_1, \ldots, \Gamma_k \in \mathcal{E}.$

Let P be a transition probability on a Polish space[1] E. By the Kolmogorov existence theorem (see Theorem 3.7), there is a probability space and a Markov chain with transition probability P. It turns out that an arbitrary stochastic dynamical system on a Polish space can be regarded as a solution of the stochastic difference equation

$$X(0) = x, \qquad X(t+1) = F(X(t), Z(t+1)), \qquad t = 0, 1, \ldots, \tag{1.1}$$

where $Z(1), Z(2), \ldots$ is a sequence of independent identically distributed random variables (i.i.d.s). In the engineering literature, a sequence of this type is called *discrete-time white noise*. We have the following representation result.

Theorem 1.2 *Let E be a Polish space, and let $\mathcal{E} = \mathcal{B}(E)$ be the family of its Borel sets. Then, for any transition probability P, there exists a measurable mapping $F\colon E \times [0, 1] \mapsto E$ such that, for any sequence of independent random variables $Z(1), Z(2), \ldots$ with uniform distribution on $[0, 1]$ and for any $x \in E$, the process X given by (1.1) is a Markov chain with transition probability P.*

Proof We follow Kifer (1986). First we construct F in the case where E is a countable set and $E = \mathbb{R}$. Let $E = \{1, 2, \ldots\} = \mathbb{N}$ and let $p_{n|m} = P(n, \{1\}) + \cdots + P(n, \{m\})$, $n, m \in \mathbb{N}$. Define $F(n, r) = m$ for $r \in [p_{n|m-1}, p_{n|m})$. Measurability is obvious. If Z has a uniform distribution on $[0, 1]$ then $\mathbb{P}(\{\omega : F(n, Z(\omega)) = m\}) = P(n, \{m\})$, as required.

If $E = \mathbb{R}$ then we first define $F_0(x, a) := P(x, (-\infty, a])$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, and set $F(x, r) := \inf\{a \colon r \le F_0(x, a)\}$, $r \in [0, 1]$. It is clear that if Z has a uniform distribution on $[0, 1]$ then $\mathbb{P}(\{\omega\colon F(x, Z(\omega)) \in \Gamma\}) = P(x, \Gamma)$ for $x \in \mathbb{R}$ and $\Gamma \in \mathcal{B}(\mathbb{R})$. By the so-called *Borel isomorphism theorem*, actually due to K. Kuratowski (see Kuratowski 1934, Dynkin and Yushkevich 1978 or Srivastava

[1] That is, a separable metric space that is complete with respect to some equivalent metric.

1998), any uncountable Polish space is measurably isomorphic to $\mathbb{R}$ and the result follows. □

Remark 1.3 The result is a version of the famous Skorokhod embedding theorem. A similar theorem, but for controlled stochastic dynamical systems, can be found in Zabczyk (1996), pp. 26–7.

Once the result is proved for complete separable metric spaces E, it can be generalized to all spaces E that are measurably isomorphic to such spaces, that is, to all Borel spaces.

If E is a linear space it is convenient to reformulate (1.1) slightly; namely, we write

$$dX(t) := X(t) - X(t-1), \qquad Y(t) := Z(1) + \cdots + Z(t),$$
$$dY(t) := Y(t) - Y(t-1)$$

and change the function F to $\widetilde{F}(x, r) := -x + F(x, r)$. Then

$$dX(t) = \widetilde{F}(X(t-1), dY(t)).$$

Considering an embedding $r \mapsto \delta_r$ of the interval [0, 1] into the space of all finite measures on [0, 1], we arrive at an equation in which the noise enters linearly. Namely, we set $G(x)\lambda := \int_{[0,1]} \widetilde{F}(x, r)\lambda(dr)$ and $\widetilde{Z}(t) := \delta_{Z(t)}$, $L(t) := \widetilde{Z}(1) + \ldots + \widetilde{Z}(t)$, $dL(t) := \widetilde{Z}(t)$. Then

$$dX(t) = G(X(t-1))\, dL(t), \qquad t = 1, 2, \ldots, \tag{1.2}$$

and the increments of L are independent. Thus the diffusion operator G acts on the increments of the noise in a linear way. An analogous result holds for continuous-time stochastic dynamical systems with values in $\mathbb{R}^d$; see Section 1.5. The main ideas come from Itô (1951).

1.2 Deterministic continuous-time systems

Deterministic continuous-time dynamical systems are families $(F_t,\ t \geq 0)$ of transformations from a given state space E into E satisfying the semigroup property $F_t F_s = F_{t+s}$, $t, s \geq 0$. The trajectory starting from x is the mapping $X(t) = F_t(x)$ of the parameter t. Are the dynamical systems always solutions of differential equations? The answer is obviously no! Differential equations are well defined only on rather special state spaces. Even if we assume that the state space is $E = \mathbb{R}^d$ and that the transformation $(t, x) \mapsto F_t(x)$ is continuous, there are still dynamical systems not defined by differential equations, as the following example shows.

Example 1.4 Consider a continuous but nowhere differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ and define

$$F_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + t \\ f(x+t) + y - f(x) \end{pmatrix}, \qquad t \geq 0,\ x, y \in \mathbb{R}.$$

Then the trajectories of F_t are nowhere differentiable and consequently (F_t) cannot be a solution to an equation of the form $\mathrm{d}F_t(z)/\mathrm{d}t = A(F_t(z))$, where $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

However, if $E = \mathbb{R}^d$ and all trajectories of a given dynamical system are continuously differentiable then they are solutions of the ordinary differential equation $\mathrm{d}X(t)/\mathrm{d}t = A(X(t))$, where $A(x) := \lim_{t \downarrow 0}(1/t)(F_t(x) - x)$.

If E is an infinite-dimensional space then the answer can again be positive provided that the flow is not too pathological.

Example 1.5 Assume that E is a Banach space and that, for each $t \geq 0$, F_t is a continuous linear transformation on E and, for each $x \in E$, $t \mapsto F_t(x)$ is a continuous mapping. Then in a proper sense $\mathrm{d}F_t(x)/\mathrm{d}t = A(F_t(x))$, $t > 0$, where A is usually an unbounded linear operator on E. In fact (F_t) is a C_0-semigroup and A is its generator; see Chapter 9 and Appendix B. Often we write $F_t = \mathrm{e}^{At}$.

Example 1.6 Assume that E is a Hilbert space, that the transformations F_t are *contractions*, i.e. $|F_t(x) - F_t(y)| \leq |x - y|$ for $t \geq 0$ and $x, y \in E$, and that, for each x, $t \mapsto F_t(x)$ is a continuous function. Then $\mathrm{d}F_t(x)/\mathrm{d}t = A(F_t(x))$, $t > 0$, where A is a so-called dissipative, usually unbounded and non-linear, operator; see Chapter 10.

A differential equation does not always uniquely determine the flow of its solutions. There are many subtleties here and interesting results; see for instance Hartman (1964).

1.3 Stochastic continuous-time systems

In analogy to discrete-time dynamical systems, a *stochastic continuous-time dynamical system* is a family (P_t) of stochastic kernels $P_t(x, \Gamma)$, $t \geq 0$, $x \in E$, $\Gamma \in \mathcal{E}$. We interpret $P_t(x, \Gamma)$ as the probability that the system will be in a set Γ at time t, provided that its initial position is x. More precisely, we have the following definition.

Definition 1.7 A family of probability measures $P_t(x, \cdot)$ on E is said to be a *transition probability* if: