Undergraduate Texts in Mathematics

Larry Smith

Linear Algebra

Second Edition

$$\begin{pmatrix} \overbrace{\begin{matrix} e_1 - t & & \\ & e_1 - t & \\ & & \ddots \\ & & & e_1 - t \end{matrix}}^{n_1} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\begin{matrix} e_k - t & & \\ & e_k - t & \\ & & \ddots \\ & & & e_k - t \end{matrix}}^{n_k} \end{pmatrix}$$

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With 21 Figures

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Larry Smith
Mathematisches Institut
Universität Göttingen
D3400 Göttingen
West Germany

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continued after Index

Preface

This text is written for a course in linear algebra at the (U.S.) sophomore undergraduate level, preferably directly following a one-variable calculus course, so that linear algebra can be used in a course on multidimensional calculus. Realizing that students at this level have had little contact with complex numbers or abstract mathematics, the book deals almost exclusively with real finite-dimensional vector spaces in a setting and formulation that permits easy generalization to abstract vector spaces. The parallel complex theory is developed in the exercises.

The book has as a goal the principal axis theorem for real symmetric transformations, and a more or less direct path is followed. As a consequence there are many subjects that are not developed, and this is intentional.

However, a wide selection of examples of vector spaces and linear transformations is developed, in the hope that they will serve as a testing ground for the theory. The book is meant as an *introduction* to linear algebra and the theory developed contains the essentials for this goal. Students with a need to learn more linear algebra can do so in a course in abstract algebra, which is the appropriate setting. Through this book they will be taken on an excursion to the algebraic/analytic zoo, and introduced to some of the animals for the first time. Further excursions can teach them more about the curious habits of some of these remarkable creatures.

For the second edition of the book I have added, amongst other things, a safari into the wilderness of canonical forms, where the hardy student can persue the Jordan form with the tools developed in the preceding chapters.

Göttingen,
June 1984

LARRY SMITH

Contents

Vectors in the plane and space 1

In physics certain quantities such as *force*, *displacement*, *velocity*, and *acceleration* possess both a magnitude and a direction and they are most usually represented geometrically by drawing an arrow with the magnitude and direction of the quantity in question. Physicists refer to the arrow as a *vector*, and call the quantities so represented *vector quantities*. In the study of the calculus the student has no doubt encountered vectors, and their algebra, particularly in connection with the study of lines and planes and the differential geometry of space curves. Vectors can be described as *ordered pairs* of points (**P**, **Q**) which we call the *vector from P to Q* and often denote by $\overrightarrow{\mathbf{PQ}}$. This is substantially the same as the physics definition, since all it amounts to is a technical description of the word "arrow." **P** is called the *initial point* and **Q** the *terminal point*.

For our purposes it will be convenient to regard two vectors as being equal if they have the same length and the same magnitude. In other words we will regard $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{RS}}$ as determining the *same vector* if $\overrightarrow{\mathbf{RS}}$ results by moving $\overrightarrow{\mathbf{PQ}}$ parallel to itself.

(*N.B.* Vectors that conform to this definition are called *free vectors*, since we are "free to pick" their initial point. Not all "vectors" that occur in nature conform to this convention. If the vector quantity depends not only on its direction and magnitude but its initial point it is called a *bound vector*. For example, torque is a bound vector. In the force-vector diagram represented by Figure 1.1 $\overrightarrow{\mathbf{PQ}}$ does not have the same effect as $\overrightarrow{\mathbf{RS}}$ in pivoting a bar. In this book we will consider only free vectors.)

With this convention of equality of vectors in mind it is clear that if we *fix a point* **O** *in space* called *the origin*, then we may regard all our vectors as having their initial point at **O**. The vector $\overrightarrow{\mathbf{OP}}$ will very often be abbreviated to $\vec{\mathbf{P}}$, if the point **O** which serves as the origin of all vectors is clear from

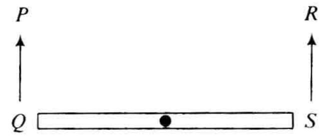

Figure 1.1

context. The vector $\vec{\mathbf{P}}$ is called the *position vector* of the point **P** relative to the origin **O**.

In physics vector quantities such as force vectors are often added together to obtain a resultant force vector. This process may be described as follows. **Suppose an origin O has been fixed**. Given vectors $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ their sum is defined by the Figure 1.2. That is, draw the parallelogram determined by the three points **P**, **O** and **Q**. Let **R** be the fourth vertex and set $\vec{\mathbf{P}} + \vec{\mathbf{Q}} = \vec{\mathbf{R}}$.

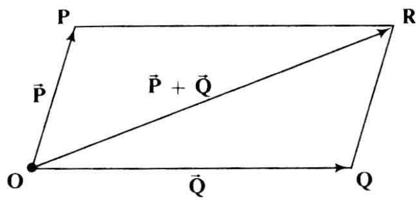

Figure 1.2

The following basic rules of vector algebra may be easily verified by elementary Euclidean geometry.

(1) $\vec{\mathbf{P}} + \vec{\mathbf{Q}} = \vec{\mathbf{Q}} + \vec{\mathbf{P}}$.
(2) $(\vec{\mathbf{P}} + \vec{\mathbf{Q}}) + \vec{\mathbf{R}} = \vec{\mathbf{P}} + (\vec{\mathbf{Q}} + \vec{\mathbf{R}})$.
(3) $\vec{\mathbf{P}} + \vec{\mathbf{O}} = \vec{\mathbf{P}} = \vec{\mathbf{O}} + \vec{\mathbf{P}}$.

It is also possible to define the operation of multiplying a vector by a number. Suppose we are given a vector $\vec{\mathbf{P}}$ and a number a. If $a > 0$ we let $a\vec{\mathbf{P}}$ be the vector with the same direction as $\vec{\mathbf{P}}$ only a times as long (see Figure 1.3). If $a < 0$ we set $a\vec{\mathbf{P}}$ equal to the vector of magnitude a times the magnitude of $\vec{\mathbf{P}}$ but having direction *opposite* of $\vec{\mathbf{P}}$ (see Figure 1.4). If $a = 0$ we set $a\vec{\mathbf{P}}$

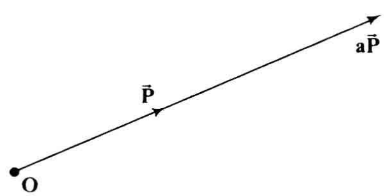

Figure 1.3

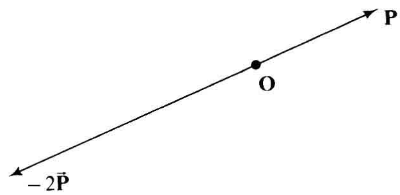

Figure 1.4

equal to $\vec{\mathbf{O}}$. It is then easy to show that vector algebra satisfies the following additional rules:

(4) $\vec{\mathbf{P}} + (-1\vec{\mathbf{P}}) = \vec{\mathbf{O}}$

(5) $a(\vec{\mathbf{P}} + \vec{\mathbf{Q}}) = a\vec{\mathbf{P}} + a\vec{\mathbf{Q}}$

(6) $(a + b)\vec{\mathbf{P}} = a\vec{\mathbf{P}} + b\vec{\mathbf{P}}$

(7) $(ab)\vec{\mathbf{P}} = a(b\vec{\mathbf{P}})$

(8) $0\vec{\mathbf{P}} = \vec{\mathbf{O}}$, $1\vec{\mathbf{P}} = \vec{\mathbf{P}}$

Note that Rule 6 involves two types of addition, namely addition of numbers and addition of vectors.

Vectors are particularly useful in studying lines and planes in space. Suppose that an origin **O** has been fixed and L is the line through the two points **P** and **Q** as in Figure 1.5. Suppose that **R** is any other point on L.

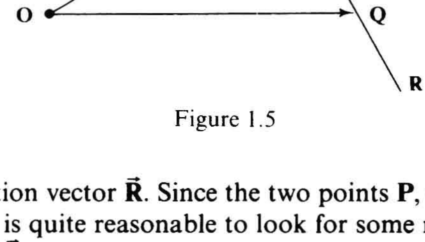

Figure 1.5

Consider the position vector $\vec{\mathbf{R}}$. Since the two points **P**, **Q** completely determine the line L, it is quite reasonable to look for some relation between the vectors $\vec{\mathbf{P}}$, $\vec{\mathbf{Q}}$, and $\vec{\mathbf{R}}$. One such relation is provided by Figure 1.6. Observe that

$$\vec{\mathbf{S}} + \vec{\mathbf{P}} = \vec{\mathbf{Q}}$$

Therefore if we write $-\vec{\mathbf{P}}$ for $(-1)\vec{\mathbf{P}}$ we see that

$$\vec{\mathbf{S}} = \vec{\mathbf{Q}} - \vec{\mathbf{P}}.$$

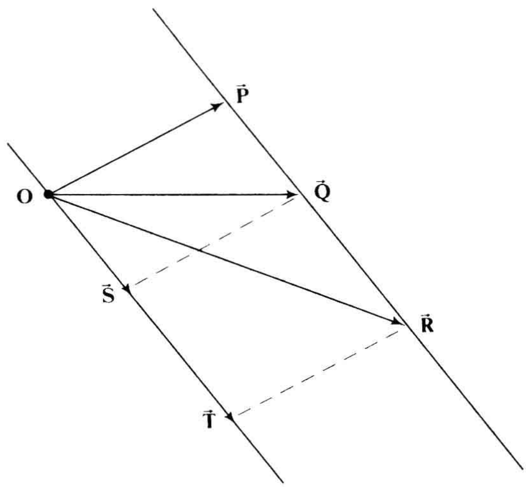

Figure 1.6

Notice that there is a number t such that

$$\vec{\mathbf{T}} = t\vec{\mathbf{S}}.$$

Moreover

$$\vec{\mathbf{R}} = \vec{\mathbf{P}} + \vec{\mathbf{T}}$$

and hence we find

$$(*) \qquad \vec{\mathbf{R}} = \vec{\mathbf{P}} + t(\vec{\mathbf{Q}} - \vec{\mathbf{P}}).$$

Equation (*) is called the *vector equation* of the line L. To make practical computations with this equation it is convenient to introduce in addition to the origin **O** a cartesian coordinate system as in Figure 1.7. Every point **P** then has coordinates (x, y, z), and if we have two points **P** and **Q** with coordinates $(x_{\mathbf{P}}, y_{\mathbf{P}}, z_{\mathbf{P}})$ and $(x_{\mathbf{Q}}, y_{\mathbf{Q}}, z_{\mathbf{Q}})$ then it is quite easy to check that

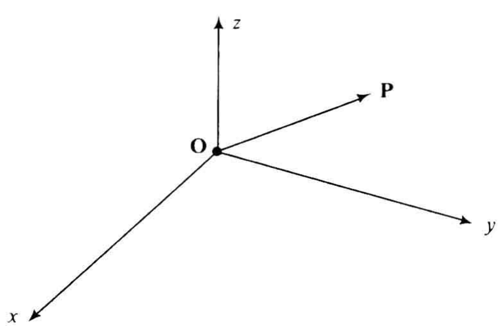

Figure 1.7

$\vec{\mathbf{P}} + \vec{\mathbf{Q}}$ is the position vector of the point with components $(x_{\mathbf{P}} + x_{\mathbf{Q}}, y_{\mathbf{P}} + y_{\mathbf{Q}}, z_{\mathbf{P}} + z_{\mathbf{Q}})$. Likewise for a number a the vector $a\vec{\mathbf{P}}$ is the position vector of the point with coordinates $(ax_{\mathbf{P}}, ay_{\mathbf{P}}, az_{\mathbf{P}})$. Thus we find by considering the coordinates of the points represented Equation (*) that (x, y, z) lies on the line L through **P**, **Q** iff

$$(**)\qquad \begin{aligned} x &= x_{\mathbf{P}} + t(x_{\mathbf{Q}} - x_{\mathbf{P}}),\\ y &= y_{\mathbf{P}} + t(y_{\mathbf{Q}} - y_{\mathbf{P}}),\\ z &= z_{\mathbf{P}} + t(z_{\mathbf{Q}} - z_{\mathbf{P}}). \end{aligned}$$

EXAMPLE 1. Does the point (1, 2, 3) lie on the line passing through the points (4, 4, 4) and (1, 0, 1)?

Solution. Let L be the line through $\mathbf{P} = (4, 4, 4)$ and $\mathbf{Q} = (1, 0, 1)$. Then the points of L must satisfy the equations

$$\begin{aligned} x &= 4 + t(1 - 4) = 4 - 3t,\\ y &= 4 + t(0 - 4) = 4 - 4t,\\ z &= 4 + t(1 - 4) = 4 - 3t, \end{aligned}$$

where t is a number. Let us check if this is possible:

$$\begin{aligned} 1 &= 4 - 3t,\\ 2 &= 4 - 4t,\\ 3 &= 4 - 3t. \end{aligned}$$

The first equation gives

$$-3 = -3t \qquad t = 1.$$

Putting this in the last equation gives

$$3 = 4 - 3 = 1$$

which is impossible. Therefore (1, 2, 3) does not lie on the line through (4, 4, 4) and (1, 0, 1).

EXAMPLE 2. Let L_1 be the line through the points (1, 0, 1) and (1, 1, 1). Let L_2 be the line through the points (0, 1, 0) and (1, 2, 1). Determine if the lines L_1 and L_2 intersect. If so find their point of intersection.

Solution. The equations of L_1 are

$$\begin{aligned} x &= 1 + t_1(1 - 1) = 1,\\ y &= 0 + t_1(1 - 0) = t_1,\\ z &= 1 + t_1(1 - 1) = 1. \end{aligned}$$

The equations of L_2 are

$$\begin{aligned} x &= 0 + (1 - 0)t_2 = t_2,\\ y &= 1 + (2 - 1)t_2 = 1 + t_2,\\ z &= 0 + (1 - 0)t_2 = t_2. \end{aligned}$$

If a point lies on both of these lines we must have

$$\begin{aligned} 1 &= t_2, \\ t_1 &= 1 + t_2, \\ 1 &= t_2. \end{aligned}$$

Therefore $t_2 = 1$ and $t_1 = 2$. Hence (1, 2, 1) is the only point these lines have in common.

EXAMPLE 3. Determine if the lines L_1 and L_2 with equations

$$L_1 \quad \begin{aligned} x &= 1 - 3t, \\ y &= 1 + 3t, \\ z &= t, \end{aligned}$$

$$L_2 \quad \begin{aligned} x &= -2 - 3t, \\ y &= 4 + 3t, \\ z &= 1 + t, \end{aligned}$$

have a point in common.

Solution. If a point (x, y, z) lies on both lines it must satisfy both sets of equations, so there is a number t_1 such that

$$\begin{aligned} x &= 1 - 3t_1, \\ y &= 1 + 3t_1, \\ z &= t_1, \end{aligned}$$

and a number t_2 with

$$\begin{aligned} x &= -2 - 3t_2, \\ y &= 4 + 3t_2, \\ z &= 1 + t_2, \end{aligned}$$

and the answer to the problem is reduced to determining if in fact two such numbers can be found, that is if the simultaneous equations

$$(*) \qquad \begin{aligned} 1 - 3t_1 &= -2 - 3t_2, \\ 1 + 3t_1 &= 4 + 3t_2, \\ t_1 &= 1 + t_2, \end{aligned}$$

have any solutions. Writing these equations in the more usual form they become

$$\begin{aligned} 3 &= 3t_1 - 3t_2, \\ -3 &= -3t_1 + 3t_2, \\ -1 &= -t_1 + t_2. \end{aligned}$$

By dividing the first equation by 3, the second by -3, and multiplying the third by -1 we get

$$
\begin{aligned}
1 &= t_1 - t_2,\\
1 &= t_1 - t_2,\\
1 &= t_1 - t_2,
\end{aligned}
$$

giving

$$t_1 = 1 + t_2.$$

What does this mean? It means that no matter what value of t_2 we choose there is a value of t_1, namely $t_1 = 1 + t_2$, which satisfies Equations (*). By varying the values of t_2 we get all the points on the line L_2. For each such value of t_2 the fact that there is a (corresponding) value of t_1 solving Equations (*) shows that every point of the line L_2 lies on the line L_1. Therefore these lines must be the same!

The lesson to be learned from this example is that the equations of a line are not unique. This should be geometrically clear since we only used two points of the line to determine the equations, and there are many such possible pairs of points.

EXAMPLE 4. Determine if the lines L_1 and L_2 with equations

$$
L_1 \quad
\begin{aligned}
x &= 1 + t,\\
y &= 1 + t,\\
z &= 1 - t,
\end{aligned}
$$

$$
L_2 \quad
\begin{aligned}
x &= 2 + t,\\
y &= 2 - t,\\
z &= 2 - t,
\end{aligned}
$$

have a point in common.

Solution. As in Example 3 our task is to determine if the simultaneous equations

$$
(*) \qquad
\begin{aligned}
1 + t_1 &= 2 + t_2,\\
1 + t_1 &= 2 - t_2,\\
1 - t_1 &= 2 - t_2,
\end{aligned}
$$

has any solutions. In more usual form these equations become

$$
\begin{aligned}
-1 &= -t_1 + t_2,\\
-1 &= -t_1 - t_2,\\
-1 &= t_1 - t_2.
\end{aligned}
$$

Adding the first two equations gives

$$-2 = -2t_1,$$

so t_1 must equal 1. Putting this into the last equation we get

$$-1 = 1 - t_2,$$

so t_2 must equal 2. But substituting these values of t_1 and t_2 into either of the first two equations leads to a contradiction, namely

$$-1 = -1 + 2 = 1,$$
$$-1 = -1 - 2 = -3,$$

therefore no values of t_1 and t_2 can simultaneously satisfy Equations (*) so the lines have no point in common.

In Chapter 13 we will take up the study of solving simultaneous linear equations in detail. There we will explain various techniques and "tests" that will make the problems encountered in Examples 3 and 4 routine.

Suppose now that **P**, **Q**, and **R** are three noncolinear points. Then they determine a unique plane Π. If we introduce a fixed origin **O** then it is possible to deduce an equation that is satisfied by the position vectors of points of Π. Considering Figure 1.8 shows that

$$\vec{\mathbf{A}} - \vec{\mathbf{Q}} = s(\vec{\mathbf{P}} - \vec{\mathbf{Q}}) + t(\vec{\mathbf{R}} - \vec{\mathbf{Q}})$$

that is

$$(*) \qquad \vec{\mathbf{A}} = s(\vec{\mathbf{P}} - \vec{\mathbf{Q}}) + t(\vec{\mathbf{R}} - \vec{\mathbf{Q}}) + \vec{\mathbf{Q}}.$$

Equation (*) is called the *vector equation* of the plane Π. Compare it to the vector equation of a line. Note the presence of the two parameters s and t instead of the single parameter t

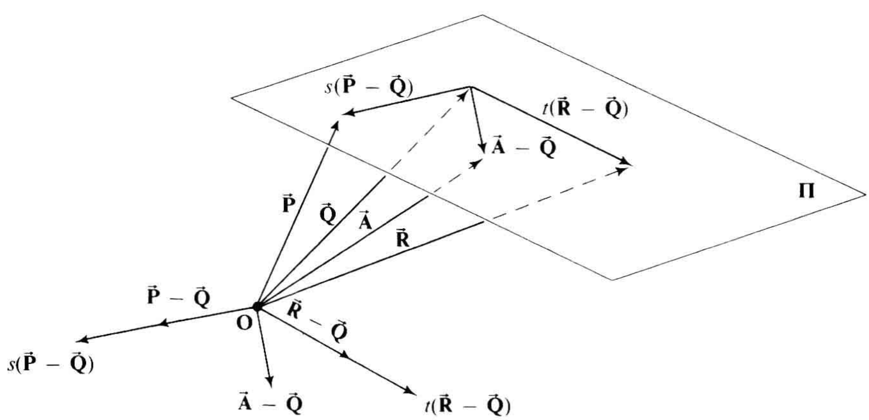

Figure 1.8

If we now introduce a coordinate system and pass to components in Equation (∗) we obtain:

$$(**)\qquad \begin{aligned} x &= s(x_{\mathbf{P}} - x_{\mathbf{Q}}) + t(x_{\mathbf{R}} - x_{\mathbf{Q}}) + x_{\mathbf{Q}}, \\ y &= s(y_{\mathbf{P}} - y_{\mathbf{Q}}) + t(y_{\mathbf{R}} - y_{\mathbf{Q}}) + y_{\mathbf{Q}}, \\ z &= s(z_{\mathbf{P}} - z_{\mathbf{Q}}) + t(z_{\mathbf{R}} - z_{\mathbf{Q}}) + z_{\mathbf{Q}}. \end{aligned}$$

We may regard Equation (∗∗) as the equation of the plane Π or we may regard it as a system of three equations in the two unknowns s, t which we may formally eliminate and obtain the more familiar equation

$$(**)\qquad ax + by + cz + d = 0$$

where we *may* take (or twice these values, or −7 times, etc.)

$$\begin{aligned} a &= (y_{\mathbf{R}} - y_{\mathbf{Q}})(z_{\mathbf{P}} - z_{\mathbf{Q}}) - (z_{\mathbf{R}} - z_{\mathbf{Q}})(y_{\mathbf{P}} - y_{\mathbf{Q}}), \\ b &= (z_{\mathbf{R}} - z_{\mathbf{Q}})(x_{\mathbf{P}} - x_{\mathbf{Q}}) - (x_{\mathbf{R}} - x_{\mathbf{Q}})(z_{\mathbf{P}} - z_{\mathbf{Q}}), \\ c &= (x_{\mathbf{R}} - x_{\mathbf{Q}})(y_{\mathbf{P}} - y_{\mathbf{Q}}) - (y_{\mathbf{R}} - y_{\mathbf{Q}})(x_{\mathbf{P}} - x_{\mathbf{Q}}), \\ d &= -(ax_{\mathbf{P}} + by_{\mathbf{P}} + cz_{\mathbf{P}}). \end{aligned}$$

Equation (∗∗) is also called the equation of the plane Π.

Example 5. Find the equation of the plane through the points

$$(1, 0, 1), \qquad (0, 1, 0), \qquad (1, 1, 1).$$

Determine if the point (0, 0, 0) lies in this plane.

Solution. We know that the equation has the form

$$ax + by + cz + d = 0$$

and all we must do is crank out values for a, b, c, d. (Remember they are not unique.) We must have

$$\begin{aligned} a + c + d &= 0, \\ b + d &= 0, \\ a + b + c + d &= 0, \end{aligned}$$

since the points (1, 0, 1), (0, 1, 0), and (1, 1, 1) lie in this plane. Thus

$$a + c = 0, \qquad d = 0, \qquad b = 0, \qquad a = -c.$$

So the plane has the equation

$$x - z = 0$$

and (0, 0, 0) lies in it.

Example 6. Determine the equation of the line of intersection of the planes

$$\begin{aligned} x - z &= 0, \\ x + y + z + 1 &= 0. \end{aligned}$$

Solution. The line in question has an equation of the form

$$x = a + ut,$$
$$y = b + vt,$$
$$z = c + wt,$$

for suitable numbers a, b, c, u, v, w. Since such points must lie in both planes we have

$$a + ut - (c + wt) = 0,$$
$$a + ut + b + vt + c + wt + 1 = 0,$$

for all values of t. Put $t = 0$. Then

$$a - c = 0,$$
$$a + b + c + 1 = 0.$$

The first equation yields $a = c$. Combining this with the second equation and setting $b = 1$ yields $2a + 2 = 0$. Hence $a = -1 = c$. Next put $t = 1$. Then

$$0 = a + ut - (c + wt) = -1 + u + 1 - w,$$
$$0 = a + ut + b + vt + c + wt + 1$$
$$= -1 + u + 1 + v - 1 + w + 1.$$

The first equation yields $u = w$. Combining this with the second equation and setting $u = 1$ yields $w = u = 1$ and $v = -2$. Then

$$x = -1 + t,$$
$$y = 1 - 2t,$$
$$z = -1 + t,$$

are the equations of a line containing the two points $(-1, 1, -1)$ and $(0, -1, 0)$ which lie in both planes and hence must be the line of intersection.

Exercises

1. Suppose that an origin **O** and a coordinate system have been fixed. Let **P** be a point. Define vectors $\vec{\mathbf{E}}_1$, $\vec{\mathbf{E}}_2$, and $\vec{\mathbf{E}}_3$ by requiring that they be the position vectors of the points (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively. Let the coordinates of **P** be $(x_{\mathbf{P}}, y_{\mathbf{P}}, z_{\mathbf{P}})$. Show that

$$\vec{\mathbf{P}} = x_{\mathbf{P}}\vec{\mathbf{E}}_1 + y_{\mathbf{P}}\vec{\mathbf{E}}_2 + z_{\mathbf{P}}\vec{\mathbf{E}}_3.$$

The vectors

$$x_{\mathbf{P}}\vec{\mathbf{E}}_1, \qquad y_{\mathbf{P}}\vec{\mathbf{E}}_2, \qquad z_{\mathbf{P}}\vec{\mathbf{E}}_3$$

are called the *component vectors* of $\vec{\mathbf{P}}$ relative to the given coordinate system.