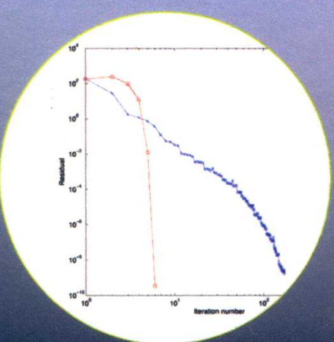
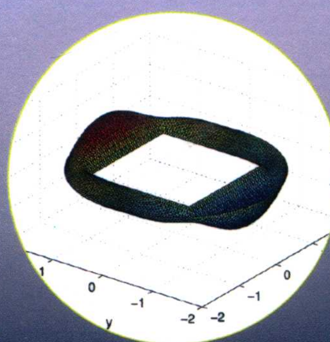
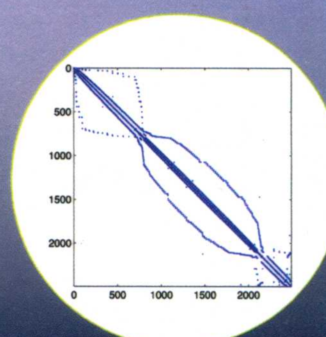
Pure and Applied Mathematics:
A Wiley Series of Texts, Monographs, and Tracts

NUMERICAL ANALYSIS *of* PARTIAL DIFFERENTIAL EQUATIONS







S.H. LUI

WILEY

Numerical Analysis of Partial Differential Equations

S. H. Lui
Department of Mathematics
University of Manitoba
Winnipeg, Manitoba, Canada

WILEY
A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Lui, S. H. (Shaun H.), 1961–
Numerical analysis of partial differential equations / S.H. Lui.
p. cm. — (Pure and applied mathematics, a Wiley-Interscience series of texts, monographs, and tracts)
Includes bibliographical references and index.
ISBN 978-0-470-64728-8 (hardback)
1. Differential equations, Partial—Numerical solutions. I. Title.
QA377.L84 2011
518'.64—dc23 2011013570

Printed in the United States of America.

oBook ISBN: 978-1-118-11113-0
ePDF ISBN: 978-1-118-11110-9
ePub ISBN: 978-1-118-11111-6

10 9 8 7 6 5 4 3 2 1

Numerical Analysis of Partial Differential Equations

PURE AND APPLIED MATHEMATICS

A Wiley Series of Texts, Monographs, and Tracts

A complete list of the titles in this series appears at the end of this volume.

PREFACE

This is a book on the numerical analysis of partial differential equations (PDEs). This beautiful subject studies the theory behind algorithms used to approximate solutions of PDEs. The ultimate goal is to design methods which are accurate and efficient. It is a relatively young field which draws upon powerful theory from many branches of mathematics, both pure and applied.

The contents in this book are suitable for a two-semester course at the senior undergraduate or beginning graduate level, for students in mathematical sciences and engineering. The emphasis is on elliptic PDEs, with one chapter discussing evolutionary PDEs. A prerequisite for reading this book is a solid undergraduate course in analysis. Some exposure to numerical analysis and PDEs is helpful.

Our aim is to offer an introduction to most of the important concepts in the numerical analysis of PDEs. The two-dimensional Poisson equation is the model problem upon which the various methods are analyzed. Because of the simplicity of this equation, the analysis can almost always be carried out in full. In this sense, this book is entirely self-contained; the student does not need to consult other books for the proof of a theorem. The only exception is the chapter on the mathematical theory of elliptic PDEs, where very few proofs are given. Many students taking this course may have no prior experience in this area and thus this chapter is simply an introduction to the topic by examples. A proper treatment of PDEs requires several semesters, which most students cannot accommodate in their programs. Some may argue that there is too much emphasis on the Poisson equation, however, our response is that for an introductory course, this equation is appropriate because it gives a good picture of the kinds of results expected without the complications involved in more general PDEs. Convection-diffusion equations and nonlinear equations are of course interesting, but they belong to more advanced courses and many topics there are still under active research. Other omitted topics include finite volume methods, discontinuous Galerkin methods, meshless methods, Monte Carlo methods, wavelets, eigenvalue problems, inverse problems, free boundary value problems, etc. Implementation and visualization issues are not discussed at all.

The topics covered are the three main discretization methods of elliptic PDEs: finite difference, finite elements and spectral methods. These are presented in Chapters 1, 3 and 5, respectively. In between are discussions on the mathematical theory of elliptic PDEs in Chapter 2 and numerical linear algebra in Chapter 4. Time-dependent PDEs make a brief appearance in Chapter 6. Multigrid and domain decomposition, are covered in Chapters 7 and 8. These are among the most efficient techniques for solving PDEs today. Chapter 9 contains a discussion of PDEs posed on infinite domains. The main issue here is how to pose the boundary condition on the artificial boundary which is necessary on a finite computational domain. Methods for nonlinear problems are briefly described in Chapter 10. Here, we also describe some important nonlinear problems in many fields of science and engineering. These can serve as computing projects for students from different disciplines.

Each chapter can be, and have been, expanded by other authors into a course by itself! Most chapters can be covered in approximately 10 hours. The exceptions are: Chapter 3 (finite elements) and Chapter 5 (spectral methods) which require about 15 hours each to cover all sections; Chapter 6 (multigrid), Chapter 9 (infinite domains) and Chapter 10 (nonlinear problems) need about five hours each.

A few words about the ordering of the chapters are called for. The material on the finite difference method requires few prerequisites and thus is placed in the first chapter. The analysis of the finite element and spectral methods uses the language of Sobolev spaces and their properties, which are conveniently covered in the second chapter. Having seen the structured matrices in the finite difference method and the unstructured matrices in the finite element method, readers are well motivated to appreciate and comprehend the issues in numerical linear algebra in Chapter 4. A possible alternative ordering of the first part of the book is to discuss Sobolev spaces first, followed by the three discretization techniques and finally numerical linear algebra. The problem is that the material on PDE theory appears to many students as abstract, dry and unmotivated. Furthermore, the discussion on numerical PDEs does not begin until the fourth week. This book does not have to be read in the order presented but Chapter 2 should precede all subsequent chapters except 4, 9 and 10. Chapters 6 through 9 can be read in any order, but they rely heavily on material from the first four chapters.

No one learns mathematics by reading alone. Exercises are an integral part of this book and students are encouraged to try them. They are essential for reinforcing the material and many extend theories and techniques developed in the text. Both theoretical and programming problems are available, with the former prefixed by E and the latter prefixed by P. Answers to selected written exercises are given. Although this book does not emphasize the implementation of the algorithms, readers willing to invest time on the programming exercises will gain a much better appreciation of the subject. As already mentioned, about half of the chapters can be covered in about four weeks. The time frame can easily extend by one to two weeks per chapter if students do a substantial fraction of the written and programming exercises.

Almost all material in this book are well known to numerical analysts and have been gleaned from various sources listed in the bibliography. References to texts or monographs are generally given in place of the original articles. There are already excellent texts on each of the areas discussed in this book but there does not appear to be one which covers all the topics here. The present book should serve as a good preparation for more advanced work. Readers are encouraged to send their comments and corrections to luish@cc.umanitoba.ca. A webpage http://home.cc.umanitoba.ca/~luish/numpde has been created for this book. It will contain errata as well as some MATLAB programs.

ACKNOWLEDGMENTS

I have learned a great deal from the books and papers of many mathematicians (a partial list appears in the References) and whose ideas I have followed in this book. In fact, I was studying several topics for the first time as I was writing this book–experts in the fields should have no difficulty pointing out my level of ignorance. I sincerely thank my colleagues Susanne Brenner, Raymond Chan, Qiang Du, Martin Gander, Laurence Halpern, Ronald Haynes, Felix Kwok, Jie Shen, Xue-Cheng Tai and Justin Wan for reading parts of this book on short notice and making many insightful suggestions. This book is poorer because I have not the time, energy and/or expertise to implement all their proposed changes. I take this opportunity to thank the many students who have taken courses based on earlier drafts of this book. They pointed out many typos, inaccuracies and made numerous suggestions which have immeasurably improved the presentation. Of course, I am responsible for all remaining mistakes in the book. It has truly been a humbling and rewarding experience having so many wonderful students in my classes.

Michael Doob, Amy Hendrickson and Jonas Lippuner have been extremely helpful with the finer points of LaTeX. I also thank my editor Susanne Steitz-Filler and the staff at John Wiley & Sons, Inc. Finally, financial support from NSERC and the University of Manitoba are gratefully acknowledged.

CONTENTS

CHAPTER 1

FINITE DIFFERENCE

The finite difference method was one of the first numerical methods used to solve partial differential equations (PDEs). It replaces differential operators by finite differences and the PDE becomes a (finite) system of equations. Its simplicity and ease of computer implementation make it a popular choice for PDEs defined on regular geometries. One drawback is that it becomes awkward when the geometry is not regular or cannot be mapped to a regular geometry. Another disadvantage is that its error analysis is not as sharp as that of the other methods covered in this book (finite element and spectral methods). A recurring theme is that the analysis of, and properties of, discrete operators mimic those of the differential operators. Examples include integration by parts, maximum principle, energy method, Green's function and the Poincaré–Friedrichs inequality.

1.1 SECOND-ORDER APPROXIMATION FOR Δ

Consider the Poisson equation

$$-\Delta u = f \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian in two dimensions. Throughout this chapter, except the sections on polar coordinates and curved boundaries, Ω is the unit square $(0,1)^2 = \{(x,y),\ 0 < x, y < 1\}$.

The Poisson equation is a fundamental equation which arises in elasticity, electromagnetism, fluid mechanics and many other branches of science and engineering. Because an explicit expression of the solution is available only in a few exceptional cases, we often rely on numerical methods to approximate the solution. The goal of the subject of numerical analysis of PDEs is to design a numerical method which approximates the solution accurately and efficiently. Roughly speaking, a method is accurate if the computed solution differs from the exact solution by an amount which goes to zero as h, a discretization parameter, goes to zero. A method is efficient if the amount of computation and storage requirement of the method do not grow quickly as a function of the size of the input data, f, in the case of the Poisson equation. The remaining pages of this book will be preoccupied with these issues and will culminate in algorithms (multigrid and domain decomposition) which are optimal in the sense that the amount of work to approximate the solution is no more than a linear function of the size of the input.

We take a uniform grid of size $h = 1/n$, where n is a positive integer. Let

$$\Omega_h = \{x_{ij} = (ih, jh),\ \ 1 \le i, j \le n-1\}$$

denote the set of interior grid points and

$$\partial\Omega_h = \{(0, jh), (1, jh), (jh, 0), (jh, 1),\ \ 1 \le j \le n-1\}$$

denote the boundary grid points. [Observe that the corner points $(0,0), (1,0), (0,1), (1,1)$ are not in $\partial\Omega_h$.] Define $\overline{\Omega}_h = \Omega_h \cup \partial\Omega_h$, which consists of $(n+1)^2 - 4$ points.

The finite difference method seeks the solution of the PDE at the grid points in Ω. Specifically, the $(n-1)^2$ unknowns are $u_{ij} = u(ih, jh),\ 1 \le i, j \le n-1$. We obtain $(n-1)^2$ equations by approximating the differential equation by a finite difference approximation at each interior grid point. That is,

$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2} = f_{ij} := f(x_{ij}).$$

(These equations will be derived later when we discuss the consistency of the scheme.) The boundary values are $u_{0j} = u_{nj} = u_{i0} = u_{in} = 0,\ 1 \le i, j \le n-1$; they are known from the boundary condition. The system of linear equations is denoted by the discrete Poisson equation

$$-\Delta_h u_h = f_h.$$

Here, u_h is the vector of unknowns arranged in an order so that Δ_h is block tridiagonal:

$$u_h = [u_{11}, \cdots, u_{n-1,1}, u_{12}, \cdots, u_{n-1,2}, \cdots, u_{n-1,n-1}]^T,$$

$f_h = [f_{11}, \cdots, f_{n-1,n-1}]^T$ and

$$-\Delta_h = \frac{1}{h^2}\begin{bmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{bmatrix}, \quad T = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}.$$

Both T and I above are $(n-1) \times (n-1)$ matrices with I the identity matrix. An alternative representation of Δ_h is given by the molecule

$$\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}.$$

Before proceeding further, we define some norms which measure the size of vectors and matrices. First consider the infinity norm on $\mathbb{R}^N$. For any $x \in \mathbb{R}^N$, define

$$|x|_\infty = \max_{1 \le i \le N} |x_i|.$$

For any $N \times N$ matrix A, we claim that

$$|A|_\infty := \sup_{x \in \mathbb{R}^N \setminus 0} \frac{|Ax|_\infty}{|x|_\infty} = \max_{1 \le i \le N} \sum_{j=1}^{N} |a_{ij}|.$$

In other words, $|A|_\infty$ equals the maximum row sum of the matrix. To see this, for any $x \in \mathbb{R}^N \setminus 0$,

$$\frac{|Ax|_\infty}{|x|_\infty} = \frac{\left\| \begin{bmatrix} \sum_{j=1}^N a_{1j}x_j \\ \vdots \\ \sum_{j=1}^N a_{Nj}x_j \end{bmatrix} \right\|_\infty}{|x|_\infty} = \frac{\max_{1 \le i \le N} \left| \sum_{j=1}^{N} a_{ij}x_j \right|}{|x|_\infty} \le \max_{1 \le i \le N} \sum_{j=1}^{N} |a_{ij}|.$$

Hence $|A|_\infty \le \max_{1 \le i \le N} \sum_{j=1}^N |a_{ij}|$. To show equality, suppose $A \ne 0$ and the maximum row sum occurs on row i. Define $x_j = \text{sign}(a_{ij})$. Then $|x|_\infty = 1$ and $|Ax|_\infty = \sum_{j=1}^N |a_{ij}|$. As an application, we note that $|\Delta_h|_\infty = 8h^{-2}$.

Another useful norm is the Euclidean (or 2-) norm defined by $|x|_2^2 = x^T x$ and for matrix A,

$$|A|_2 = \sup_{x \ne 0} \frac{|Ax|_2}{|x|_2}.$$

One useful property is $|A|_2 = \sigma_1$, where σ_1^2 is the largest eigenvalue of $A^T A$. If A is symmetric, then $|A|_2 = |\lambda_1|$ while $|A^{-1}|_2 = |\lambda_N^{-1}|$, where λ_1 and λ_N are eigenvalues of A of largest and smallest magnitude, respectively.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a normed vector space X are said to be **equivalent** if there are positive constants c_i such that for every v in X, $c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1$. It is well known that any two norms on a finite-dimensional space are equivalent. This means that we can use $|\cdot|_2$ or $|\cdot|_\infty$, whichever is more convenient. Although c_1 and c_2 are independent of $v \in X$, they do depend on N, the dimension of the space.

Finally, we define $C^r(\overline{\Omega})$, for $0 \leq r \leq \infty$, as the space of r times continuously differentiable functions on $\overline{\Omega}$ with norm

$$\|v\|_{C^r(\overline{\Omega})} = \max_{0\leq|\alpha|\leq r} \sup_{x\in\overline{\Omega}} |D^\alpha v(x)|.$$

Here $D^\alpha v$ denotes any derivative of v up to rth order, where α is a multi-index. For instance, if $\alpha = (2,1)$, then $D^\alpha v$ denotes the 3rd derivative v_{xxy}. We abbreviate $C^0(\overline{\Omega})$ as $C(\overline{\Omega})$. If $v \in C^r(\overline{\Omega})$, then $D^\alpha v \in C(\overline{\Omega})$ for every α such that $|\alpha| := \alpha_1 + \alpha_2 \leq r$. Define

$$\|v\|^*_{C^r(\overline{\Omega})} = \max_{|\alpha|=r} \sup_{x\in\overline{\Omega}} |D^\alpha v(x)|. \tag{1.2}$$

This is not a norm but it comes up frequently in the analysis of finite difference schemes. In the one-dimensional case,

$$\|v\|_{C^2([0,1])} = \max_{x\in[0,1]} (|v(x)|, |v'(x)|, |v''(x)|), \qquad \|v\|^*_{C^2([0,1])} = \max_{x\in[0,1]} |v''(x)|.$$

In this book, all functions are real unless otherwise specified. Also, c appears in many places and denotes a positive constant whose value may differ in different occurrences. The same remark applies to c_1, c_2, C, C_1, C_2, etc.

Next, we shall demonstrate several properties of $-\Delta_h$. These are crucial in estimating the error of the approximate solution.

Positive Definiteness

Let $(\cdot,\cdot)$ be the L^2 inner product defined by

$$(u,v) = \int_\Omega uv$$

for all square-integrable functions u and v defined on Ω. It is well known that $-\Delta$ is a self-adjoint and positive definite operator. This means that for all smooth functions u, v, w vanishing on $\partial\Omega$ with $w \not\equiv 0$,

$$(-\Delta u, v) = (u, -\Delta v) \quad \text{and} \quad (-\Delta w, w) > 0.$$

(A rigorous justification of self-adjointness is non-trivial since it requires checking the domain of the operator.)

We now show that the discrete operator has analogous properties. By inspection, $-\Delta_h$ is symmetric. We now show that it is positive definite. This is shown in two