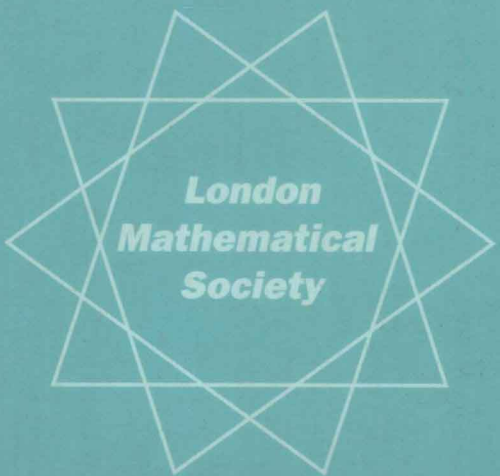
London Mathematical Society
Lecture Note Series 318

Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations

Dan Henry



AMBRIDGE
IVERSITY PRESS

London Mathematical Society Lecture Note Series. 318

Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations

Dan Henry

with editorial assistance from

Jack Hale

and

Antônio Luiz Pereira

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
40 West 20th Street, New York, NY 10011-4211, USA

www.cambridge.org
Information on this title: www.cambridge.org/9780521574914

© Cambridge University Press 2005

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2005

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data

Henry, D.
Perturbation of the boundary – value problems of partial differential equations / D. Henry.
p. cm. – (London Mathematical Society lecture note series ; 318)
Includes bibliographical references and index.
ISBN 0-521-57491-9 (pbk. : alk. paper)
1. Boundary value problems. 2. Perturbation (Mathematics) 3. Differential equations, Partial. I. Title. II. Series.
QA379.H46 2005
515'.353 – dc22 2004054767

ISBN-13 978-0-521-57491-4 paperback
ISBN-10 0-521-57491-9 paperback

Cambridge University Press has no responsibility for
the persistence or accuracy of URLs for external or
third-party Internet Web sites referred to in this book
and does not guarantee that any content on such
Web sites is, or will remain, accurate or appropriate.

Perturbation of the Boundary in Boundary-Value Problems of Partial Differential Equations

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor N. J. Hitchin, Mathematical Institute,
University of Oxford, 24-29 St Giles, Oxford OX1 3LB, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at www.cambridge.org

100 Stopping time techniques for analysts and probabilists, L. EGGHE
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
107 Compactification of Siegel moduli schemes, C.-L. CHAI
109 Diophantine analysis, J. LOXTON & A. VAN DER POORTEN (eds)
113 Lectures on the asymptotic theory of ideals, D. REES
116 Representations of algebras, P.J. WEBB (ed)
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M.-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
144 Introduction to uniform spaces, I.M. JAMES
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
148 Helices and vector bundles, A.N. RUDAKOV *et al*
149 Solitons, nonlinear evolution equations and inverse scattering, M. ABLOWITZ & P. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
155 Classification theories of polarized varieties, TAKAO FUJITA
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
171 Squares, A.R. RAJWADE
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M. FOURMAN, P. JOHNSTONE & A. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD *et al*
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
188 Local analysis for the odd order theorem, H. BENDER & G. GLAUBERMAN
189 Locally presentable and accessible categories, J. ADAMEK & J. ROSICKY
190 Polynomial invariants of finite groups, D.J. BENSON
191 Finite geometry and combinatorics, F. DE CLERCK *et al*
192 Symplectic geometry, D. SALAMON (ed)
194 Independent random variables and rearrangement invariant spaces, M. BRAVERMAN
195 Arithmetic of blowup algebras, WOLMER VASCONCELOS
196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI *et al*
198 The algebraic characterization of geometric 4-manifolds, J.A. HILLMAN
199 Invariant potential theory in the unit ball of $\mathbf{C}^n$, MANFRED STOLL
200 The Grothendieck theory of dessins d'enfant, L. SCHNEPS (ed)
201 Singularities, JEAN-PAUL BRASSELET (ed)
202 The technique of pseudodifferential operators, H.O. CORDES
203 Hochschild cohomology of von Neumann algebras, A. SINCLAIR & R. SMITH
204 Combinatorial and geometric group theory, A.J. DUNCAN, N.D. GILBERT & J. HOWIE (eds)
205 Ergodic theory and its connections with harmonic analysis, K. PETERSEN & I. SALAMA (eds)
207 Groups of Lie type and their geometries, W.M. KANTOR & L. DI MARTINO (eds)
208 Vector bundles in algebraic geometry, N.J. HITCHIN, P. NEWSTEAD & W.M. OXBURY (eds)
209 Arithmetic of diagonal hypersurfaces over finite fields, F.Q. GOUVÊA & N. YUI
210 Hilbert C^*-modules, E.C. LANCE
211 Groups 93 Galway / St Andrews I, C.M. CAMPBELL *et al* (eds)
212 Groups 93 Galway / St Andrews II, C.M. CAMPBELL *et al* (eds)
214 Generalised Euler-Jacobi inversion formula and asymptotics beyond all orders, V. KOWALENKO *et al*
215 Number theory 1992–93, S. DAVID (ed)
216 Stochastic partial differential equations, A. ETHERIDGE (ed)
217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
218 Surveys in combinatorics, 1995, PETER ROWLINSON (ed)
220 Algebraic set theory, A. JOYAL & I. MOERDIJK
221 Harmonic approximation, S.J. GARDINER

试读结束，需要全本PDF请购买 www.ertongbook.com

222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
224 Computability, enumerability, unsolvability, S.B. COOPER, T.A. SLAMAN & S.S. WAINER (eds)
225 A mathematical introduction to string theory, S. ALBEVERIO, *et al*
226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
228 Ergodic theory of $\mathbf{Z}^d$ actions, M. POLLICOTT & K. SCHMIDT (eds)
229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J.W.S. CASSELS & E.V. FLYNN
231 Semigroup theory and its applications, K.H. HOFMANN & M.W. MISLOVE (eds)
232 The descriptive set theory of Polish group actions, H. BECKER & A.S. KECHRIS
233 Finite fields and applications, S.COHEN & H. NIEDERREITER (eds)
234 Introduction to subfactors, V. JONES & V.S. SUNDER
235 Number theory 1993–94, S. DAVID (ed)
236 The James forest, H. FETTER & B. GAMBOA DE BUEN
237 Sieve methods, exponential sums, and their applications in number theory, G.R.H. GREAVES *et al*
238 Representation theory and algebraic geometry, A. MARTSINKOVSKY & G. TODOROV (eds)
240 Stable groups, FRANK O. WAGNER
241 Surveys in combinatorics, 1997, R.A. BAILEY (ed)
242 Geometric Galois actions I, L. SCHNEPS & P. LOCHAK (eds)
243 Geometric Galois actions II, L. SCHNEPS & P. LOCHAK (eds)
244 Model theory of groups and automorphism groups, D. EVANS (ed)
245 Geometry, combinatorial designs and related structures, J.W.P. HIRSCHFELD *et al*
246 p-Automorphisms of finite p-groups, E.I. KHUKHRO
247 Analytic number theory, Y. MOTOHASHI (ed)
248 Tame topology and o-minimal structures, LOU VAN DEN DRIES
249 The atlas of finite groups: ten years on, ROBERT CURTIS & ROBERT WILSON (eds)
250 Characters and blocks of finite groups, G. NAVARRO
251 Gröbner bases and applications, B. BUCHBERGER & F. WINKLER (eds)
252 Geometry and cohomology in group theory, P. KROPHOLLER, G. NIBLO & R. STÖHR (eds)
253 The q-Schur algebra, S. DONKIN
254 Galois representations in arithmetic algebraic geometry, A.J. SCHOLL & R.L. TAYLOR (eds)
255 Symmetries and integrability of difference equations, P.A. CLARKSON & F.W. NIJHOFF (eds)
256 Aspects of Galois theory, HELMUT VÖLKLEIN *et al*
257 An introduction to noncommutative differential geometry and its physical applications 2ed, J. MADORE
258 Sets and proofs, S.B. COOPER & J. TRUSS (eds)
259 Models and computability, S.B. COOPER & J. TRUSS (eds)
260 Groups St Andrews 1997 in Bath, I, C.M. CAMPBELL *et al*
261 Groups St Andrews 1997 in Bath, II, C.M. CAMPBELL *et al*
262 Analysis and logic, C.W. HENSON, J. IOVINO, A.S. KECHRIS & E. ODELL
263 Singularity theory, BILL BRUCE & DAVID MOND (eds)
264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds)
265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
267 Surveys in combinatorics, 1999, J.D. LAMB & D.A. PREECE (eds)
268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
269 Ergodic theory and topological dynamics, M.B. BEKKA & M. MAYER
270 Analysis on Lie groups, N.T. VAROPOULOS & S. MUSTAPHA
271 Singular perturbations of differential operators, S. ALBEVERIO & P. KURASOV
272 Character theory for the odd order theorem, T. PETERFALVI
273 Spectral theory and geometry, E.B. DAVIES & Y. SAFAROV (eds)
274 The Mandlebrot set, theme and variations, TAN LEI (ed)
275 Descriptive set theory and dynamical systems, M. FOREMAN *et al*
276 Singularities of plane curves, E. CASAS-ALVERO
277 Computational and geometric aspects of modern algebra, M.D. ATKINSON *et al*
278 Global attractors in abstract parabolic problems, J.W. CHOLEWA & T. DLOTKO
279 Topics in symbolic dynamics and applications, F. BLANCHARD, A. MAASS & A. NOGUEIRA (eds)
280 Characters and automorphism groups of compact Riemann surfaces, THOMAS BREUER
281 Explicit birational geometry of 3-folds, ALESSIO CORTI & MILES REID (eds)
282 Auslander-Buchweitz approximations of equivariant modules, M. HASHIMOTO
283 Nonlinear elasticity, Y. FU & R.W. OGDEN (eds)
284 Foundations of computational mathematics, R. DEVORE, A. ISERLES & E. SÜLI (eds)
285 Rational points on curves over finite, fields, H. NIEDERREITER & C. XING
286 Clifford algebras and spinors 2ed, P. LOUNESTO
287 Topics on Riemann surfaces and Fuchsian groups, E. BUJALANCE *et al*
288 Surveys in combinatorics, 2001, J. HIRSCHFELD (ed)
289 Aspects of Sobolev-type inequalities, L. SALOFF-COSTE
290 Quantum groups and Lie theory, A. PRESSLEY (ed)
291 Tits buildings and the model theory of groups, K. TENT (ed)
292 A quantum groups primer, S. MAJID
293 Second order partial differential equations in Hilbert spaces, G. DA PRATO & J. ZABCZYK
294 Introduction to the theory of operator spaces, G. PISIER
295 Geometry and Integrability, LIONEL MASON & YAVUZ NUTKU (eds.)
296 Lectures on invariant theory, IGOR DOLGACHEV
297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
299 Kleinian Groups and Hyperbolic 3-Manifolds Y. KOMORI, V. MARKOVIC, C. SERIES (eds.)
300 Introduction to Möbius Differential Geometry, UDO HERTRICH-JEROMIN
301 Stable Modules and the D(2)-Problem, F.E.A. JOHNSON
302 Discrete and Continuous Nonlinear Schrödinger Systems, M. J. ABLOWITZ, B. PRINARI, A. D. TRUBATCH
303 Number Theory and Algebraic Geometry, MILES REID ALEXEI SKOROBOGATOV (eds.)
304 Groups St Andrews 2001 in Oxford Vol. 1, COLIN CAMPBELL, EDMUND ROBERTSON, GEOFF SMITH (eds.)
305 Groups St Andrews 2001 in Oxford Vol. 2, C. M. CAMPBELL, E.F. ROBERTSON, G.C. SMITH (eds.)
307 Surveys in Combinatorics 2003, C. D. WENSLEY (ed.)
309 Corings and Comodules, TOMASZ BRZEZINSKI & ROBERT WISBAURER
310 Topics in Dynamics and Ergodic Theory, SERGEY BEZUGLYI & SERGIY KOLYADA (eds.)
312 Foundations of Computational Mathematics, Minneapolis 2002, FELIPE CUCKER *et al* (eds.)

Contents

	Introduction	*page* 1
1	**Geometrical Preliminaries**	3
1.1	Some notation	3
2	**Differential Calculus of Boundary Perturbations**	18
3	**Examples Using the Implicit Function Theorem**	27
3.1	Torsional Rigidity	27
3.2	Simple Eigenvalues of the Dirichlet Problem for the Laplacian	32
3.3	Capacity	36
3.4	Green's Function	39
3.5	Simple Eigenvalues of Robin's Problem	40
3.6	Simple Eigenvalue of a General Dirichlet Problem	44
4	**Bifurcation Problems**	46
4.1	Multiple Eigenvalues of the Dirichlet Problem for the Laplacian	46
4.2	Variation of a Turning Point	51
4.3	A Bifurcation Problem wit Two-Dimensional Kernel	53
4.4	Generic Simplicity of Eigenvalues of a Self-Adjoint 2m-Order Dirichlet Problem	55
5	**The Transversality Theorem**	60
6	**Generic Perturbation of the Boundary**	79
6.1	Generic Simplicity of Eigenvalues of the Dirichlet Problem for Δ	80

6.2	Simplicity of Eigenvalues with a Reflection-symmetry Constraint	83
6.3	Simplicity of Real Eigenvalues of a General Second-Order Dirichlet Problem	85
6.4	Generic Simplicity of Eigenvalues for the Neumann Problem for Δ	88
6.5	Generic Simplicity of $\Delta u + f(x, u, \nabla u) = 0$ in Ω, $u = 0$ on $\partial\Omega$	89
6.6	Generic Simplicity of Eigenvalues of Robin's Problem	94
6.7	Generic Simplicity of Solutions of $\Delta u + f(u, x) = 0$ in Ω, $\partial u/\partial N = g(k, u)$ on $\partial\Omega$	100
6.8	Generic Simplicity of Complex Eigenvalues of a Dirichlet Problem	106
7	**Boundary Operators for Second-Order Elliptic Equations**	114
7.1	Weakly Singular Integral Operators with $C^r_*(\alpha)$ Kernels	116
7.2	Integral Equations with $C^r_*(\alpha)$ Kernels	122
7.3	Fourier Transform and Composition of Weakly Singular Kernels	126
7.4	Limits of Singular Integrals	135
7.5	Fundamental Solutions	139
7.6	Calculation of some boundary Operators	141
8	**The Method of Rapidly-Oscillating Solutions**	152
8.1	Introduction	152
8.2	Formal Asymptotic Solutions	154
8.3	Exact Solutions	158
8.4	Example 6.8 Revisited: Generic Simplicity of Complex Eigenvalues	161
8.5	Generic Simplicity of Solutions of a System	168
Appendix 1	Eigenvalues of the Laplacian in the Presence of Symmetry	183
Appendix 2	On Micheletti's Metric Space	188
	References	199
	Index	203

Introduction

Perturbation of the boundary (or of the domain of definition of a boundary value problem) is a rather neglected mathematical topic, though it has attracted occasional interest: Rayleigh [33] in 1877 (the first edition), Hadamard [9] in 1908, Courant and Hilbert [5] in the German edition of 1937, Polya and Szëgo [31] in 1951, Garabedian and Schiffer [7] in 1952, and some more recent work [3, 14, 6, 13, 26, 21–23, 42, 35, 32, 27, 10, 11]. The list is far from complete, but is notably sparse.

There seem to be two related reasons for this neglect: (1) the subject is too easy; (2) it is too difficult. If you are interested only in Fundamental Questions, this is certainly a trivial topic. One perturbs the region by applying a diffeomorphism near the identity; but you can change variables via this diffeomorphism to keep the region fixed, and are then only perturbing the coefficients in a fixed region. It is simply the chain rule. However, if you try to carry out this trivial change of variables, you will become mired in such long and difficult calculations that you'll be tempted to quit. If you persist, and are fortunate, and are extremely careful, there may be a miraculous simplification at the end. On experiencing this miracle for the second time I became suspicious – theorems 2.2, 2.4 show how to go directly to the "miracle," bypassing the computational morass. (Peetre [27] also found part of this result, and it is implicit in Courant and Hilbert [5: vol. 1 p. 260] for variational problems.) It is, at the end, merely the chain rule, and we may then apply standard tools (implicit function theorem, Liapunov, Schmidt method, transversality theorem) to problems of perturbation of the boundary. But the standard tools are not enough – new problems arise requiring, for example, a more general form of the transversality theorem for problems with Fredholm index $-\infty$ (ch. 5). Open problems abound. The calculus developed in chapter 2 applies – in principle – to almost any boundary or initial boundary-value problem, but often leads immediately to difficult unsolved problems. To avoid excessive depression of author and reader, we will

concentrate on questions we can answer – generally boundary-value problems for scalar second-order elliptic equations. The subject is not, after all, entirely trivial.

I have worked on perturbation of the boundary sporadically since about 1973, after reading Joseph's article [13]. The formulas of theorems 2.2, 2.4 date from 1975 – in a more complicated version – and most of the examples of chapters 3 and 4 (and a few from chapter 6) were developed in 1975–1981 and exposed in seminars at the University of Kentucky, Brown University and the University of São Paulo. In 1982, I had the opportunity to develop this topic at some length [11] at the University of Brasilia. Later that year I generalized the transversality theorem (lectures at São Carlos, September 1982), which solved some problems left open in [11] and raised a host of new open problems. These demanded some tedious calculation – such as those in chapter 7 – which were only completed recently, and then only in special cases: much remains to be done. If the word "I" seems to appear excessively here, it is because very few other people have worked on these problems in the past twelve years with a comparable approach, and my work has been independent of these few (aside from some of the examples cited below). There are, of course, other notions of a "small change in the domain" besides "image under diffeomorphism near the identity" (see, for example, [3, 26, 10]); the advantage of using such regular perturbations will hopefully become more clear as we proceed.

In the past five years, my work has been supported by FAPESP, and of course by IME-USP. It would have been nice to have this ready for the fiftieth anniversary of the University of São Paulo, but as usual I missed the deadline. Still, better late than never: Happy 51st birthday!

Most of the following (Chapters 1 to 7) was written in 1985. But at the end of that year, I found a way to circumvent Horrible Chapter Seven, the method of rapidly oscillating solutions. My course was clear: everything should be rewritten with the new method! Unfortunately, I couldn't seem to find the time and energy needed for the task. Finally, Jack Hale and Antonio Luiz Pereira persuaded me to publish it as it stands, with only minor corrections. (Except that the correction of Theorem 7.6.10 is not so minor, and this required rewriting Example 6.8.) A new chapter was added, on the method of rapidly oscillating solutions, with a new example (8.5: generic simplicity of solutions of a system) to show the power of the method. And, with a few years perspective, Chapter 7 does not seem so horrible.

Thanks Jack and Antonio Luiz for getting it moving!

Chapter 1

Geometrical Preliminaries

In this chapter we introduce some notation, find convenient representations for regions in $\mathbb{R}^n$ with smooth boundary (1.2–1.7) and develop some tools needed later: approximation by smooth regions (1.8), smooth extension of functions (1.9), smooth deformations of regions (1.10), differentiation of integrals (1.11), and differential operators on a hypersurface (1.12). In short, a mixed bag of topics which will be needed later, and it may be wise to omit details until the need becomes apparent.

We treat only smoothly-bounded regions. Initial-boundary-value problems lead naturally to the study of regions with corners; but our examples will be elliptic boundary-value problems, and smooth regions provide sufficient variety.

1.1 Some Notation

For a function f defined near $x \in \mathbb{R}^n$, the m^{th} derivative at x, $D^m f(x)$, may be considered as a homogeneous polynomial of degree m ($h \mapsto D^m f(x)h^m$) on $\mathbb{R}^n$, or as a symmetric m-linear form, or as the collection of partial derivatives

$$D^m f(x) = \left\{ \left(\frac{\partial}{\partial x} \right)^\alpha f(x) : |\alpha| = m \right\},$$

depending on convenience. Then the norm $|D^m f(x)|$ may denote

$$\max_{|\alpha|=m} \left| \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right| \text{ or } \max_{|h|\leq 1} |D^m f(x)h^m|.$$

The last version has the advantage of being independent of rotation of coordinates in $\mathbb{R}^n$, but the norms are equivalent.

If Ω is an open set in $\mathbb{R}^n$ and $m \geq 0$ an integer, $C^m(\Omega)$ is the space of m-times continuously and bounded differentiable functions on Ω whose derivatives

试读结束，需要全本PDF请购买 www.ertongbook.com

extend continuously to the closure $\overline{\Omega}$, with the usual norm

$$\|\phi\|_{C^m(\Omega)} = \max_{0\le j\le m} \sup_{x\in\Omega} |D^j\phi(x)|.$$

The space of values is some normed linear space E, and is not clear which "E" is meant, we may write $C^m(\Omega, E)$.

- $C^m_{unif}(\Omega)$ is the closed surface of $C^m(\Omega)$ consisting of functions whose m^{th} derivative is uniformly continuous; if Ω is bounded, this is $C^m(\Omega)$.
- $C^{m,\alpha}(\Omega)$ is the subspace of $C^m_{unif}(\Omega)$ consisting of functions whose m^{th} derivative is Hölder continuous with exponent α $(0 < \alpha \le 1)$, provided with the norm

  $$\|\phi\|_{C^{m,\alpha}(\Omega)} = \max\left(\|\phi\|_{C^m(\Omega)}, H^\Omega_\alpha(D^m\phi)\right)$$

  where

  $$H^\Omega_\alpha(f) = \sup\{|f(x) - f(y)|/|x - y|^\alpha : x \ne y \in \Omega\}$$

  (This space is "boundedly closed" in $C^0(\Omega)$; that is, a bounded sequence in $C^{m,\alpha}(\Omega)$ which converges uniformly (in $C^0(\Omega)$) has its limits in $C^{m,\alpha}(\Omega)$.)
- $C^{m,\alpha+}(\Omega)$ is the closed subspace of $C^{m,\alpha}(\Omega)$, $0 < \alpha < 1$, consisting of functions $\phi \in C^{m,\alpha}(\Omega)$ such that

  $$|D^m\phi(x) - D^m\phi(y)|/|x - y|^\alpha \to 0 \quad \text{as } x - y \to 0 \quad (x, y \in \Omega),$$

  provided with the same $C^{m,\alpha}(\Omega)$ norm.

It is sometimes convenient to write

$$C^{m,0} \text{ for } C^m, \quad C^{m,0+} \text{ for } C^m_{unif}$$

so we allow $0 \le \alpha \le 1$ in $C^{m,\alpha}$ and $0 \le \alpha < 1$ in $C^{m,\alpha+}$. $C^{m,\alpha}_{loc}(\Omega)[C^{m,\alpha+}_{loc}(\Omega)]$ is the space of functions whose restrictions to any Ω_0 (with $\overline{\Omega_0}$ compact in Ω) are in $C^{m,\alpha}(\Omega_0)[C^{m,\alpha+}(\Omega_0)$, respectively].

$$C^\infty(\Omega) = \bigcap_m C^m(\Omega), \quad C^\infty_{loc}(\Omega) = \bigcap_m C^\infty_{loc}(\Omega)$$

$C^\omega(\Omega)$ = analytic functions defined on an open set of $\mathbb{R}^n$ containing $\overline{\Omega}$ (hence, extending to analytic functions on an open set in $\mathbb{C}^n$, when we consider $\mathbb{R}^n$ as $\{z \in \mathbb{C}^n : \operatorname{Im} z = 0\}$.)

Definition 1.2. An open set $\Omega \subset \mathbb{R}^n$ has *C^m-regular boundary* [or $C^{m,\alpha}$ or $C^{m,\alpha+}$ or $C^{m,\alpha}_{loc}$ or $C^{m,\alpha+}_{loc}$ or C^∞ or C^∞_{loc} or C^ω; regular boundary], if there exists $\phi \in C^m(\mathbb{R}^n, \mathbb{R})$ [or $C^{m,\alpha}$ or $C^{m,\alpha+}$ or ...] which is at least in $C^1_{unif}(\mathbb{R}^n, \mathbb{R})$ such that

$$\Omega = \{x \in \mathbb{R}^n : \phi(x) > 0\}$$

and $\phi(x) = 0$ implies $|\text{grad}\phi(x)| \geq 1$. We may also say Ω is a C^m region or Ω is C^m *regular* [or $C^{m,\alpha}$ or ...].

Theorem 1.3. *$\Omega \subset \mathbb{R}^n$ has C^m-regular boundary (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) – at least C^1_{unif} – if and only if there are positive constants r, M such that, given any open ball $B \subset \mathbb{R}^n$ of radius r, after appropriate rotation and translation of coordinates, we have*

$$\Omega \cap B = \{x \in \mathbb{R}^n | x_n > \psi(\hat{x})\} \cap B, \quad \hat{x} = (x_1, \ldots, x_{n-1})$$
$$\partial\Omega \cap B = \{x \in \mathbb{R}^n | x_n = \psi(\hat{x})\} \cap B$$

for some $\psi \in C^m(\mathbb{R}^{n-1}, \mathbb{R})$ (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) with norm $\leq M$. Note the conditions are trivial if $B \cap \partial\Omega = \emptyset$.

Remark. This implies easily that our definition of C^m-regular boundary is equivalent to that used by F. Browder and Agmon-Douglis-Nirenberg in their studies of elliptic boundary value problems. We will see that Def. 1.2 is very convenient for discussing perturbations of the boundary (as in 1.8 below). Our Definition 1.2 applied only when $\partial\Omega$ is at least uniformly C^1. The condition of Thm. 1.3 is more general, in that we may permit (for example) ψ to be merely Lipschitz continuous (in $C^{0,1}(\mathbb{R}^{n-1})$) which gives the "minimally smooth" domains of Stein's extension theorem [38, Sec. 6.3]. Some of our results apply to regions with convex corners, transversal intersections of smooth regions, as described in the remark following Thm. 1.9.

Proof. Suppose $\Omega = \{x \mid \phi(x) > 0\}$ is C^m (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) regular, $\phi(x) = 0 \Rightarrow |\text{grad}\ \phi(x)| \geq 1$, $L = \sup |\text{grad}\ \phi|$, and choose $r > 0$ so $|x - y| \leq 6Lr \Rightarrow |D\phi(x) - D\phi(y)| < 1/2$.

Let B be a ball of radius r in $\mathbb{R}^n$ which meets $\partial\Omega$; we may assume the center of the ball is 0. If $0 \in \partial\Omega$, choose the positive x_n-axis along the inward normal (i.e., grad ϕ (0)). Otherwise let p be a point of $\partial\Omega \cap B$ closest to 0 and choose the x_n- axis to contain p and be directed into Ω. Then $p = (0, p_n)$, $|p_n| < r$, and $\frac{\partial\phi}{\partial x_n}(p) = |D\phi(p)| \geq 1$ (possibly $p = 0$). Also $\phi(0, s)$ has the same sign as $s - p_n$ in $-r < s < r$, and for $|x - p| \leq 6Lr$ we have $\frac{\partial\phi}{\partial x_n}(x) > 1/2$.

Let $\hat{x} \in \mathbb{R}^{n-1}$, $|\hat{x}| \leq 2r$; then $|\phi(\hat{x}, x_n) - \phi(0, x_n)| \leq 2Lr$ and $\pm\phi(\hat{x}, p_n \pm 4Lr) \geq \pm\phi(0, p_n \pm 4Lr) - 2Lr > 0$ so there exists unique $\psi(\hat{x}) \in (p_n - 4Lr, p_n + 4Lr)$ with $\phi(\hat{x}, \psi(\hat{x})) = 0$. By the implicit function theorem, ψ is C^m (or $C^{m,a}$ or $C^{m,\alpha+}$, respectively) and $|D\psi(\hat{x})| = |-(\partial\phi/\partial\hat{x})/(\partial\phi/\partial x_n)| \leq 2M$ for $|\hat{x}| \leq 2r$. Choose some (fixed) $C^\infty\theta : \mathbb{R} \to [0, 1]$ with $\theta(t) = 1$ for $t \leq 1$, $\theta(t) = 0$ for $t \geq 3/2$, and let $\psi_0(\hat{x}) = \theta(|\hat{x}|/r)\psi(\hat{x})$, or zero for $|\hat{x}| \geq 3r/2$. Then $\psi_0 \in C^m(\mathbb{R}^{n-1})$ (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) with norm bounded by a multiple

(depending only on n, m, α, r) of the norm of ϕ, and

$$\Omega \cap B = \{x \in B \mid x_n > \psi_0(\hat{x})\}, \quad \partial\Omega \cap B = \{x \in B \mid x_n = \psi(\hat{x})\}.$$

For the converse, we need the following

Lemma 1.4. *Let $\sigma > \sqrt{n}, r > 0$, and for each $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let B_k be the open ball in $\mathbb{R}^n$ with radius r and center $(r/\sigma)k$, while $B_k^{1/2}$ is the concentric ball with radius $r/2$. Then every point of $\mathbb{R}^n$ is contained in some $B_k^{1/2}$, and no point is contained in more than $(2\sigma + 1)^n$ of the balls B_k.*

Proof of the Lemma. The result is invariant under a homothety ($x \mapsto cx, c =$ constant > 0) so it suffices to treat the case $r = \sigma$. If $x \in \mathbb{R}^n$ there exists $k \in \mathbb{Z}^n$ so $x - k \in [-\frac{1}{2}, \frac{1}{2})^n$, hence $|x - k| \leq \sqrt{n}/2 < \sigma/2$ so $x \in B_k^{1/2}$.

Suppose $x \in B_k$; then for each $j = 1, \ldots n, |x_j - k_j| < \sigma$ so k_j is an integer in $(x_j - \sigma, x_j + \sigma)$. But $(x_j - \sigma, x_j + \sigma)$ contains no more than $2\sigma + 1$ integers, so there are at most $(2\sigma + 1)^n$ choices of $k \in \mathbb{Z}^n$ such that $x \in B^k$.

Completion of Proof of (1.3). Assume r, M given satisfying the requirements of the theorem; we must find $\phi : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions (1.2). Choose $\sigma = \sqrt{2n+1}$ in the lemma. There is a C^∞ partition of unity $\{\phi_k\}_{k \in \mathbb{Z}^n}$

$$\operatorname{supp} \phi_k \subset B_k^{1/2}, \quad \phi_k \geq 0, \quad \sum_k \phi_k(x) \equiv 1$$

with $\|\phi_k\| C^m$ (or $\|\phi_k\| C^{m,\alpha}$) uniformly $\leq K = K(n, r, m, \alpha)$.

If $B_k \cap \partial\Omega \neq \phi$, there is a function $S_k(x)$ of class C^m(or $C^{m,\alpha}$or $C^{m,\alpha+}$) $-$ $S_k(x) = x_n - \psi(\hat{x})$ after rotation of coordinates – such that

$$\Omega \cap B_k = \{x \in B_k : S_k(x) > 0\},$$
$$\partial\Omega \cap B_k = \{x \in B_k : S_k(x) = 0\},$$

with $\|S_k\|_{C^m \text{ (or } C^{m,\alpha})} \leq M$ and

$$S_k(x) = 0 \Rightarrow |DS_k(x)| \geq 1.$$

If $B_k \subset \Omega$, let $S_k = 1$; if $B_k \cap \overline{\Omega} = \emptyset$, let $S_k = -1$. Define $\psi(x) = \sum_k \phi_k(x) S_k(x)$. Then ψ is in $C^m(\mathbb{R}^n, \mathbb{R})$ [or $C^{m,\alpha}$or $C^{m,\alpha+}$], $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$, $\psi < 0$ outside $\overline{\Omega}$. If $x \in \partial\Omega$ then $\phi_k(x) S_k(x) = 0$ for each k and at x, grad $(\phi_k S_k) = \phi_k$ grad S_k which either vanishes or has the direction of the inward normal so $|\text{grad } \psi(x)| = \sum_k \phi_k(x) |\text{ grad } S_k(x)| \geq 1$.

The following "normal coordinates" are sometimes useful.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ have C^m-regular boundary (or $C^{m,\alpha}$ or $C^{m,\alpha+}$, $2 \le m \le \infty$). There exists $r > 0$ so that if*

$$B_r(\partial\Omega) = \{x : \operatorname{dist}(x, \partial\Omega) < r\}$$
$$\pi(x) = \text{the point of } \partial\Omega \text{ nearest to } x$$
$$t(x) = \pm\operatorname{dist}(x, \partial\Omega) \quad (\text{“+” outside, “−” inside})$$

then $t(\cdot) : B_r(\partial\Omega) \to (-r, r)$, $\pi(\cdot) : B_r(\partial\Omega) \to \partial\Omega$ are well-defined, π is a C^{m-1} (or $C^{m-1,\alpha}$ or $C^{m-1,\alpha+}$) retraction onto $\partial\Omega$ ($\pi(x) = x$ when $x \in \partial\Omega$) and t has the same smoothness as $\partial\Omega$ (C^m or $C^{m,\alpha}$ or $C^{m,\alpha+}$). Further

$$x \mapsto (t(x), \pi(x)) : B_r(\partial\Omega) \to (-r, r) \times \partial\Omega$$

is a C^{m-1} (or $C^{m-1,\alpha}$ or $C^{m-1,\alpha+}$) diffeomorphism with inverse

$$(t, \xi) \mapsto \xi + tN(\xi) : (-r, r) \times \partial\Omega \to B_r(\partial\Omega)$$

where $N(\xi)$ is the unit outward normal to $\partial\Omega$ at ξ.

$$t(\cdot) \text{ is the unique solution of } |\nabla t(x)| = 1 \text{ in } B_r(\partial\Omega),$$

with $t = 0$ on $\partial\Omega$, $\partial t/\partial N > 0$ on $\partial\Omega$.

Extending the normal field N to a neighborhood of $\partial\Omega$ by

$$N(\xi + tN(\xi)) = N(\xi) \quad -r < t < r,$$

we have $N(x) = \operatorname{grad} t(x)$ on $B_r(\partial\Omega)$. Also $K(x) = DN(x) = D^2t(x)$, restricted to the tangent space at $x \in \partial\Omega$, is the curvature of $\partial\Omega$. It is sometimes convenient to call $K(x)$ the curvature, though it is degenerate ($K(x)N(x) = 0$) in the normal direction.

Remark. The fact that $t(\cdot)$ has the same smoothness as $\partial\Omega$ – does not lose a derivative, as happens with $\pi(\cdot)$ – seems to have been noted first by Gilbarg and Trudinger [8].

The best (largest) choice of r is $r = 1/\max|k|$, where k is the sectional curvature of the boundary in any (tangent) direction at any point of $\partial\Omega$.

Corollary 1.6. *A C^m-regular region $\Omega \subset \mathbb{R}^n$, $m \ge 2$, may be represented by $\{x|\phi(x) > 0\}$ where ϕ is C^m and $|\nabla\phi(x)| \equiv 1$ on a neighborhood of $\partial\Omega$. In this case, ϕ is unique on a neighborhood of $\partial\Omega$.*

Proof of (1.5). By the inverse function theorem, the C^{m-1} map $(t, \xi) \mapsto \xi + tN(\xi) = x$ has a C^{m-1} inverse $t = t(x)$, $\xi = \pi(x)$, on some neighborhood of $\partial\Omega$. On the other hand, for each $x \in \mathbb{R}^n$, $\xi \mapsto \frac{1}{2}|x - \xi|^2$ ($\xi \in \partial\Omega$) has a minimum and if ξ is a minimizing point then $x - \xi \perp T_\xi(\partial\Omega)$ so $x = \xi + tN(\xi)$

for some real t with $t = \pm\text{dist}(x, \partial\Omega)$. We show, for some $r > 0$, that $x = \xi + tN(\xi), t = \pm\text{dist}(x, \partial\Omega)$ has a unique solution ξ whenever $x \in B_r(\partial\Omega)$ so $\xi = \pi(x)$ is the (unique) nearest points.

Suppose $\Omega = \{x : \phi(x) > 0\}$, $\phi(x) = 0 \Rightarrow |\nabla\phi(x)| \geq 1$ and $r = 1/\sup|D^2\phi(x)|$. If $\text{dist}(x, \partial\Omega) < r$ but there are two "nearest points" $\xi_1 \neq \xi_2$, $x = \xi_j + tN(\xi_j)$, then $t^2 = |\xi_1 - \xi_2 + tN(\xi_1)|^2 = |\xi_1 - \xi_2|^2 + 2t(\xi_1 - \xi_2)\cdot N(\xi_1) + t^2$. Now $\phi(\xi_2) = \phi(\xi_1) = 0 = \nabla\phi(\xi_1)\cdot(\xi_2 - \xi_1) + \frac{1}{2}\int_0^1 D^2\phi(\theta\xi_2 + (1-\theta\xi_1))\cdot(\xi_2 - \xi_1)^2 d\theta$ and $N(\xi_1) = -\nabla\phi(\xi_1)/|\nabla\phi(\xi_1)|$, so for $|t| < r$

$$0 < |\xi_1 - \xi_2|^2 = 2tN(\xi_1)\cdot(\xi_2 - \xi_1) \leq |t|\sup|D^2\phi|\,|\xi_1 - \xi_2|^2 < |\xi_1 - \xi_2|^2,$$

a contradiction. Thus $t(\cdot), \pi(\cdot)$ are well-defined and C^{m-1} on $B_r(\partial\Omega)$.

Extend N to be constant on normal lines, so $N(x) = N(\pi(x))$ is C^{m-1} on $B_r(\partial\Omega)$. It is clear that $t(x + sN(x)) = t(x) + s$ when $t(x)$ and $t(x) + s$ are in $(-r, r)$, so

$$\partial t(x)/\partial x_j = \partial_j t(x + sN(x)) + s\sum_{k=1}^{n}\partial_k t(x + sN(x))\partial_j N_k(x).$$

Let $s = -t(x)$, so $x + sN(x) = \pi(x) \in \partial\Omega$, and note $\nabla t(\pi(x)) = N(\pi(x)) = N(x)$. Then on $B_r(\partial\Omega)$

$$\partial_j t(x) = N_j(x) - t(x)\sum_{k=1}^{m} N_k(x)\partial_j N_k(x) = N_j(x)$$

or $N(x) = \text{grad}\, t(x)$. (Observe that $K(x)N(x) = \partial N/\partial N(x) = 0$.) Clearly N is C^{m-1} so $t(\cdot)$ is C^m.

To identify $DN(\xi) = D^2 t(\xi)$ as the curvature of $\partial\Omega$ at ξ, we may choose coordinates so Ω is locally $\{x_n > \psi(\hat{x})\}$ where $\psi(0) = 0$, $D\psi(0) = 0$ and $K_{ij} = \partial^2\psi/\partial x_i\partial x_j\,(0)\,(1 \leq i, j \leq n-1)$ is the curvature matrix (on the tangent plane $\mathbb{R}^{n-1} \times 0$). It is easy to show $N(x) = (0, -1) + (K\hat{x}, 0) + o(|x|)$ as $x \to 0$ so $DN(0) = \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the restriction in the tangent plane is the curvature K of $\partial\Omega$ at 0.

Remark. If S is a C^2 hypersurface, $x_0 \in S$, the usual existence theory [5], [6] for first order scalar PDEs says there is a unique C^1 solution of $|\nabla t| = 1$ near x_0 with $t = 0$, $\partial t/\partial N > 0$ on S near x_0. If S is not C^2 (or $C^{1,1}$) there may be no C^1 solution of this problem. For example if $1 < p < 2$, $S = \{x : x_2 = |x_1|^p\}$, there are two "nearest points" to x when $x_1 = 0$, $x_2 > 0$ (near 0) and the gradient of $\text{dist}(x, S)$ has a discontinuous jump as x_1 crosses 0 with $x_2 > 0$.

Theorem 1.7. *Let Γ be a compact subgroup of the orthogonal group $O(n)$ and let $\Omega \subset \mathbb{R}^n$ be a C^m-regular region such that $\gamma(\Omega) = \Omega$ for all $\gamma \in \Gamma$. Then there exists C^m $\phi : \mathbb{R}^n \to \mathbb{R}$ such that for $\Omega = \{x : \phi(x) > 0\}$,*

$\phi(x) = 0 \Rightarrow |\nabla\phi(x)| \geq 1$, *and* $\phi(\gamma \cdot x) = \phi(x)$ *for all* $\gamma \in \Gamma, x \in \mathbb{R}^n$. *If* $\partial\Omega$ *is at least* C^2, *we may choose such* ϕ *with* $|\nabla\phi| = 1$ *on a neighborhood of* $\partial\Omega$.

Proof. There exists a function ϕ_0 satisfying all requirements except perhaps Γ-invariance. Let $\phi(x)$ be the average of $\gamma \mapsto \phi_0(\gamma x)$ with respect to Haar measure in Γ; then ϕ is certainly C^m and Γ-invariant. Further $x \in \Omega \Rightarrow \gamma x \in \Omega \Rightarrow \phi_0(\gamma x) > 0$ for all $\gamma \in \Gamma$ so $\phi(x) > 0$ in Ω; and similarly $\phi(x) = 0$ on $\partial\Omega$, $\phi(x) < 0$ outside $\overline{\Omega}$. On $\partial\Omega$, $N(x) = -\nabla\phi_0(x)/|\nabla\phi_0(x)|$ and it follows easily that $N(\gamma x) = \gamma N(x)$ for $x \in \partial\Omega$, $\gamma \in \Gamma$, so

$$\operatorname{grad}(\phi_0(\gamma x)) = \gamma(\operatorname{grad}\phi_0(\gamma x) = -N(x)|\operatorname{grad}\phi_0(\gamma x)|$$

and (averaging with respect to γ) $|\operatorname{grad}\phi(x)| = \operatorname{average}_\gamma|\operatorname{grad}\phi_0(\gamma x)| \geq 1$ for $x \in \partial\Omega$. If in fact $|\nabla\phi_0(x)| = 1$ near $\partial\Omega$, then $\phi(x) = \phi_0(x) = \pm\operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$ so $|\nabla\phi(x)| = 1$ near $\partial\Omega$.

A common technique in analysis is to approximate a given function by a smooth function, to facilitate calculations, and take limits only at the end. Similarly we may approximate a given region by smooth regions. If $\Omega = \{x : \phi(x) > 0\}$ is a C^m_{unif} region, and ψ is C^m-close to ϕ, we show $\{x : \psi(x) > 0\}$ is C^m-close to Ω, that is, it is diffeomorphic to Ω by a diffeomorphism C^m-close to the identity.

Theorem 1.8. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be (at least) uniformly* C^1, $\phi(x) = 0 \Rightarrow |\operatorname{grad}\phi(x)| \geq 1$, $\Omega = \{x : \phi(x) > 0\}$, *and for some* $a > 0$,

$$|\phi(x)| \geq \min\left\{\frac{1}{2}\operatorname{dist}(x, \partial\Omega), a\right\} \quad \textit{for all } x.$$

The last condition may always be achieved by modifying ϕ *away from* $\partial\Omega$. *Also, let* $r_0 > 0$.

Then there exists $\epsilon_0 > 0$ such that, if $\|\psi - \phi\|_{C^1(\mathbb{R}^n)} < \epsilon_0$, there is a diffeomorphism $h(\cdot, \psi) : \mathbb{R}^n \to \mathbb{R}^n$ supported in $B_{r_0}(\partial\Omega)$ – i.e., $h(x, \psi) = x$ when $\operatorname{dist}(x, \partial\Omega) \geq r_0$ – such that $h(\Omega; \psi) = \{x : \psi(x) > 0\}$ and $\|h(\cdot, \psi) - id_{\mathbb{R}^n}\|_{C^1(\mathbb{R}^n)} \to 0$ as $\|\psi - \phi\|_{C^1(\mathbb{R}^n)} \to 0$. If ψ is of class $C^{m,\alpha}$ [or $C^{m,\alpha+}$ or C^∞ or C^ω] then $\{x : \psi(x) > 0\}$ is a $C^{m,\alpha}$ [or $C^{m,\alpha+}$ or C^∞ or C^ω] regular region, assuming $\|\psi - \phi\|_{C^1} < \epsilon_0$.

If ϕ, ψ are $C^{m,\alpha+}$ then $h(\cdot, \psi)$ is a $C^{m,\alpha+}$ diffeomorphism, $\|h(\cdot, \psi) - id_{\mathbb{R}^n}\|_{C^{m,\alpha}} \to 0$ as $\|\psi - \phi\|_{C^{m,\alpha}} \to 0$ and $(x, \psi) \mapsto h(x, \psi) : \mathbb{R}^n \times C^{m,\alpha+}(\mathbb{R}^n) \to \mathbb{R}^n$ is $C^{m,\alpha+}$.

If ϕ, ψ are $C^{m,\alpha}$ then $h(\cdot, \psi)$ is a $C^{m,\alpha}$ diffeomorphism and $(x, \psi) \mapsto h(x, \psi) : \mathbb{R}^n \times C^{m,\alpha} \to \mathbb{R}^n$ is $C^{m,\alpha}$. As $\|\psi - \phi\|_{C^1} \to 0$ with $\|\psi\|_{C^{m,\alpha}}$ bounded, $h(\cdot, \psi)$ remains bounded in $C^{m,\alpha}$ and converges to $id_{\mathbb{R}^n}$ in $C^{k,\beta}$ when $k + \beta < m + \alpha$.

试读结束，需要全本PDF请购买 www.ertongbook.com