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Threshold Graphs and Related Topics

MONOGRAPH

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NORTH-HOLLAND

THRESHOLD GRAPHS AND RELATED TOPICS

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THRESHOLD GRAPHS AND RELATED TOPICS

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Dedicated to my mother
Prasuna
who was my first teacher
and to my father
Nadimpalli V. Subrahmanyam
who inspired me to do mathematics

NVRM

Dedicated to my dear mother Malka
and in memory of my father Ze'ev
who gave me all that I have

UNP

לאמי היקרה מלכה בת חיים ואסתר
ויד לזכר אבי ע"ה זאב בן ישעיהו ועמליה
כל מה שלי — משלהם הוא

אורי-נתן פלד

Preface

Threshold graphs have a beautiful structure and possess many important mathematical properties such as being the extreme cases of certain graph properties. They also have applications in many areas such as computer science and psychology. Their many characterizations can be relaxed in different directions to obtain new and important classes of graphs. For this reason, interest in these graphs gained momentum during last 20 years. In 1980, Golumbic devoted a chapter to them in his book *Algorithmic Graph Theory and Perfect Graphs* [Gol80]. Since then many new results related to this topic were discovered, and by now more than 100 articles have been published in various fields. Even as we started to write this book, significant results were discovered as late as this summer. We believe that this subject will continue to attract much attention and there is a need for a coherent presentation of the existing results to serve as a reference.

In writing this book, we unified several scattered results and rewrote some proofs, occasionally giving new ones. Because of space considerations we could not include every result related to threshold graphs, and we had to exercise our personal bias. In particular, we cover very little from the vast fields of hypergraphs and Boolean functions. This book is self-contained, except for very few places where we use known results from the literature. However, some chapters assume general background from linear programming or complexity theory. We tried to organize the book as much as possible so that every chapter could be studied independently after Chapter 1 and in some cases Chapter 2. Occasionally, we repeated some definitions for the convenience of the reader. We included many open problems and research ideas to make the book attractive to graduate students and researchers interested in graph theory.

We typeset this book using LaTeX 2_ε under emTeX, and we thank Eberhard Mattes for making the excellent emTeX system available to the general

public. The pictures were prepared with a combination of TEXCAD and PICTEX. These tools enabled us to produce a camera-ready copy very easily and efficiently.

Several people read parts of the manuscript and gave valuable comments. They include Waleed Al-Jasem, Srinivasa Arikati, Chris Brown, Kristine Cirino, Yee-Hong Lui, François Margot, Thomas Raschle, Ron Shamir, Andrea Sterbini, and Julin Wu. We owe them a debt of gratitude for their help. We are grateful for the kind support of the Swiss Federal Institute of Technology in Lausanne for inviting the first author to give seminars on the topics of this book, and of the Department of Mathematics at Northeastern University and the Mathematics, Statistics, and Computer Science Department of the University of Illinois at Chicago for enabling us to spend time together to accomplish what was not possible with e-mail alone. We thank Arjen Sevenster of Elsevier Science for his help in publishing this manuscript. Special thanks are due to Peter Hammer, who introduced both of us to the subject of threshold graphs, invited us to write this book and encouraged us throughout this project.

Our deepest thanks go to our wives Aparna and Ofra and our children Maya, Shilpa, Tsoni and Benny for their sacrifices and endurance through all these long years. Without their constant love and support we could not have completed this undertaking.

June 1995

Contents

Basic Terminology

We present here the basic terminology and notations used throughout the book. Additional definitions and notations are introduced in the book as needed.

For sets A and B, $A \subseteq B$ indicates that A is a subset of B, whereas $A \subset B$ indicates that A is a proper subset of B. Let $\mathcal{C}$ be a collection of sets. A set $A \in \mathcal{C}$ is *maximum* if $|A| \geq |B|$ for all $B \in \mathcal{C}$ and *minimum* if $|A| \leq |B|$ for all $B \in \mathcal{C}$. The collection $\mathcal{C}$ is *nested* if for every two sets in $\mathcal{C}$, one is a subset of the other.

A *poset* (partially ordered set) is a pair $(P, \geq)$, where P is a set and $\geq$ is a reflexive, antisymmetric and transitive relation on P. If $x \geq y$ and $x \neq y$ hold, we write $x > y$. If $x \geq y$ or $y \geq x$, x and y are *comparable*. Otherwise, x and y are *incomparable*, and we denote this condition by $x \parallel y$. A poset with no incomparable elements is said to be *total*. A *chain* is a set of mutually comparable elements, and an *antichain* is a set of mutually incomparable elements. The *Dilworth Theorem* states that the largest cardinality of an antichain equals the smallest cardinality of a set of chains partitioning P. An element x is *maximal* if there is no element y such that $y > x$. Similarly, x is *minimal* if there is no element y such that $x > y$. An element x *covers* an element y if $x > y$ and there is no z such that $x > z > y$. The *Hasse diagram* of a finite poset is a drawing where each element is represented by a point, and if x covers y, x is drawn above y and is joined to it by a line.

A *preorder* is a pair $(P, \gtrsim)$, where P is a set and $\gtrsim$ is a reflexive, transitive relation on P. If $x \gtrsim y$ and $y \gtrsim x$, we denote this condition by $x \sim y$. If $x \gtrsim y$ and $y \not\gtrsim x$, we denote this condition by $x > y$. The terms comparable, incomparable, total, chain, antichain, and maximal and minimal element are defined for preorders as for posets, and the Dilworth Theorem carries through.

The set of real numbers is denoted by $\mathbb{R}$, and we put $\mathbb{R}^+ = \{x \in \mathbb{R} : x >$

0} and $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$.

For a real-valued function f, we use the notation

$$f(S) = \sum_{s \in S} f(s).$$

A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is called a *Boolean function.* For reals $a_1, \ldots, a_n$ and t, the Boolean function f defined by

$$f(x_1, \ldots, x_n) = 0 \Longleftrightarrow \sum_{i=1}^{n} a_i x_i \leq t$$

is called a *threshold function.*

The *support* of a vector $(x_1, \ldots, x_n)$ is the set $\{i : x_i \neq 0\}$. If $S \subseteq \{a_1, \ldots, a_n\}$, then the *characteristic vector* of S is the vector $(x_1, \ldots, x_n)$ given by

$$x_i = \begin{cases} 1, & \text{if } a_i \in S \\ 0, & \text{otherwise.} \end{cases}$$

For a real x, $\lfloor x \rfloor$ denotes the largest integer $i \leq x$ and $\lceil x \rceil$ denotes the smallest integer $i \geq x$.

A *graph* G is an ordered pair (V, E), where $V = V(G)$ is a set of elements called *vertices*, $E = E(G)$ is a set of elements called *edges*, and each edge is an unordered pair of vertices (its *ends* or *end-vertices* or *end-points*). If the two ends are the same, then the edge is called a *loop*. Note that we do not allow parallel edges, i.e., all edges are distinct. The graph is said to be *finite* if V is a finite set. Unless otherwise indicated, all graphs considered from now on are finite and loopless. A graph with n vertices and e edges is referred to as an *(n, e)-graph*.

An edge with ends a, b is denoted by ab. Two vertices a and b are *adjacent* (or *neighbors*) if ab is an edge, and *non-adjacent* (or *non-neighbors*) otherwise. In the latter case, ab is a *nonedge*. If a vertex is an end of an edge, then they are *incident*. Two edges are *adjacent* if they share a common end. The *adjacency matrix* (a_{ij}) of a graph is defined by

$$a_{ij} = \begin{cases} 1, & \text{if the } i\text{-th and } j\text{-th vertices are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the *edge-vertex incidence matrix* (b_{ij}) is defined by

$$b_{ij} = \begin{cases} 1, & \text{if the } i\text{-th edge and } j\text{-th vertex are incident} \\ 0, & \text{otherwise.} \end{cases}$$

The *neighborhood* $N_G(v)$ of a vertex v in a graph G is the set of all neighbors of v, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. When G is understood, we omit the subscript G. For subsets A and B of V,

$$N_A(v) = N(v) \cap A, \qquad N_A(B) = \bigcup_{v \in B} N_A(v).$$

Similarly, $\overline{N}(v)$ denotes the set of all non-neighbors of v, and $\overline{N}_A(v) = \overline{N}(v) \cap A$. A vertex is *isolated* if its neighborhood is empty, and is *dominating* if its closed neighborhood is the entire set of vertices. We define a binary relation $\succsim$ on V by $a \succsim b \iff N[a] \supseteq N(b)$. This relation is a preorder and is called the *vicinal preorder* of G.

A graph $H = (W, F)$ is a *subgraph* of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. We then say that G *contains* H. If $W = V$, then H is a *spanning subgraph* of G. If F is the set of all edges in E with both ends in W, then H is the subgraph of G *induced* by W, and is denoted by $G[W]$. The set of edges of $G[W]$ is denoted by $E(W)$. Similarly, if W is the set of all ends of edges in F, then H is the subgraph *induced* by F. Also, $G - W = G[V - W]$ and $G - F = (V, E - F)$.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that for all $a, b \in V_1$ we have $ab \in E_1 \iff f(a)f(b) \in E_2$; in other words, G_1 and G_2 are two *labelings* of the same graph. We denote this condition by $G_1 \simeq G_2$. A property of graphs that is preserved under isomorphism is called a *graph property*. A graph property P is *hereditary* if, whenever a graph has property P, all its induced subgraphs also have property P.

The *degree* of a vertex v is $\deg(v) = |N(v)|$. The *degree sequence* of a graph with vertices $v_1, \ldots, v_n$ is $d = (\deg(v_1), \ldots, \deg(v_n))$. Every graph with the degree sequence d is a *realization* of d. A degree sequence is *unigraphic* if all its realizations are isomorphic. It is *strongly unigraphic* if there is a unique graph (V, E) with $V = \{1, \ldots, n\}$ and $\deg(i) = d_i$ for all i. A graph is a *unigraph* if its degree sequence is unigraphic. A *strong unigraph* is defined similarly. For a graph property P, a degree sequence is *potentially P-graphic* if it has a realization with property P, and *forcibly P-graphic* if all its realizations have property P. The *degree of an edge* is the unordered pair of degrees of its ends. The terms *edge-degree sequence*, *edge-unigraph* etc. are defined similarly. For more information on degree sequences, see [TCC88a, TCC88b, TCC89, Rao81].

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, F)$ such that $ab \in F \iff ab \notin E$ for each pair a, b of distinct vertices. The graph $G = (V, E)$ is *edgeless* if $E = \varnothing$. The complement of an edgeless graph is a *complete graph.* A subset W of V is a *stable set* (or an *independent set*) if $G[W]$ is an edgeless graph, and is a *clique* if $G[W]$ is a complete graph. A k*-clique* is a clique of size k. A *proper coloring* (or simply a *coloring*) of G is a partition of V into stable sets, called *color classes.* A *clique partition* of G is a partition of V into cliques. The size $\alpha(G)$ of a maximum stable set is the *stability number* of G, the size $\omega(G)$ of a maximum clique is the *clique number* of G, the size $\chi(G)$ of a minimum coloring is the *chromatic number* of G, and the size $\kappa(G)$ of a minimum clique partition is the *clique cover number* of G. A subset W of V is a *vertex cover* if $V - W$ is a stable set. The size $\rho(G)$ of a minimum vertex cover is the *vertex cover number* of G.

A *path* P_n is a graph of the form (V, E), where $V = \{1, \ldots, n\}$ and $E = \{12, 23, \ldots, (n-1)n\}$. We say that P_n is a path *joining* 1 and n. For $n \geq 3$, a *cycle* C_n is a graph of the form (V, E), where $V = \{1, \ldots, n\}$ and $E = \{12, 23, \ldots, (n-1)n, n1\}$. A graph G is *connected* if for every two vertices a and b, G contains a path joining a and b; G is *disconnected* otherwise. A maximal connected subgraph of a graph (with respect to graph containment) is a *connected component.* A graph G is *2-connected* if every two vertices belong to a cycle of G. A maximal 2-connected subgraph is a *block.*

A *matching* in a graph is a set of mutually non-adjacent edges. A vertex is *saturated* by a matching if it is an end of one of its edges, and is *unsaturated* (or *missed*) otherwise. A matching is *perfect* if it saturates every vertex of the graph.

A *bipartite graph* is a 2-colorable graph. We indicate a 2-coloring of a bipartite graph by $(A, B; E)$, where A and B are the color classes and E is the set of edges. *Hall's Theorem* states that a bipartite graph $(A, B; E)$ has a matching saturating every vertex of A if and only if every subset X of A satisfies $|X| \leq |N(X)|$.

The *union* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. If V_1 and V_2 are disjoint, we call this a *disjoint union.* When V_1 and V_2 are disjoint, the *join* of G_1 and G_2 is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E_{12})$, where E_{12} is the set of all unordered pairs consisting of a vertex of V_1 and a vertex of V_2.

Let H be a fixed graph. A graph is H*-free* if it contains no induced subgraph isomorphic to H. The disjoint union of m copies of H is denoted by