Complex Variables and Applications

James Ward Brown
Ruel V. Churchill

Sixth Edition

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ABOUT THE AUTHORS

James Ward Brown is Professor of Mathematics at The University of Michigan-Dearborn. He earned his A.B. in Physics at Harvard University and his A.M. and Ph.D. in Mathematics at The University of Michigan in Ann Arbor, where he was an Institute of Science and Technology Predoctoral Fellow. He was coauthor with Dr. Churchill of the fifth edition of *Fourier Series and Boundary Value Problems.* A past director of a research grant from the National Science Foundation, he is the recipient of a Distinguished Teaching Award from his institution, as well as a Distinguished Faculty Award from the Michigan Association of Governing Boards of Colleges and Universities. He is listed in *Who's Who in America.*

Ruel V. Churchill is Late Professor of Mathematics at The University of Michigan, where he began teaching in 1922. He received his B.S. in Physics from the University of Chicago and his M.S. in Physics and Ph.D. in Mathematics from The University of Michigan. He was coauthor with Dr. Brown of the recent fifth edition of *Fourier Series and Boundary Value Problems,* a classic text that he first wrote over fifty years ago. He was also the author of *Operational Mathematics,* now in its third edition. Throughout his long and productive career, Dr. Churchill held various offices in the Mathematical Association of America and in other mathematical societies and councils.

To the memory of my father,
GEORGE H. BROWN,
and of my long-time friend and coauthor,
RUEL V. CHURCHILL.
These distinguished men of science for years influenced
the careers of many people, including myself.

J.W.B.

PREFACE

This book is a revision of the fifth edition, published in 1990. That edition has served, just as the first four editions did, as a textbook for a one-term introductory course in the theory and applications of functions of a complex variable. This edition preserves the basic content and style of the earlier editions, the first two of which were written by the late Ruel V. Churchill alone.

In this edition, the main changes appear in the first eight chapters. To mention some of the major improvements, the chapter on residues and poles in the last edition is now divided into two chapters, one on the theory of residues and one on applications of residues. The applications chapter contains a substantial amount of new material on the use of residues in finding inverse Laplace transforms, and the material on indented contours has now been brought out of the exercises and given more emphasis. This chapter also contains a completely rewritten section on the argument principle, which was deferred until the final chapter in the earlier editions of the book. In fact, all of the material in the final chapter of the earlier editions now appears in various places throughout the present edition. The proofs of Taylor's and Laurent's theorems have been improved, and the development of properties of power series has been completely revised.

As for certain other improvements, the section on multiplication and division of power series has been enhanced pedagogically, and the discussion of values and Cauchy principal values of improper integrals has been made clearer. Finally, exercises appear more frequently, and there is a substantial number of new figures.

As was the case with the earlier editions, the *first objective* of this edition is to develop those parts of the theory that are prominent in applications of the subject. The *second objective* is to furnish an introduction to applications of residues and conformal mapping. Special emphasis is given to the use of conformal mapping in solving boundary value problems that arise in studies of heat conduction, electrostatic potential, and fluid flow. Hence the book may be considered as a companion volume to the authors' "Fourier Series and Boundary Value Problems" and Ruel V. Churchill's "Operational Mathematics," in which other classical methods for solving boundary value problems in partial differential equations are developed. The latter book also contains applications of residues in connection with Laplace transforms.

The material in the first ten chapters of this book, with various substitutions from the remaining chapters, has for many years formed the content of a three-hour course given each term at The University of Michigan. The classes have consisted mainly of seniors and graduate students majoring in mathematics, engineering, or one of the physical sciences. Before taking the course, the students have completed at least a three-term calculus sequence, a first course in ordinary differential equations, and sometimes a term of advanced calculus. In order to accommodate as wide a range of readers as possible, there are footnotes referring to texts that give proofs and discussions of the more delicate results from calculus that are occasionally needed. Some of the material in the book need not be covered in lectures and can be left for students to read on their own. If mapping by elementary functions and applications of conformal mapping are desired earlier in the course, one can skip to Chapters 8, 9, and 10 immediately after Chapter 3 on elementary functions.

Most of the basic results are stated as theorems or corollaries, followed by examples and exercises illustrating those results. A bibliography of other books, many of which are more advanced, is provided in Appendix 1. A table of conformal transformations useful in applications appears in Appendix 2.

Each copy of this new edition will be packaged with a computer diskette containing an abbreviated version of $f(z)$—The Complex Variable Program, produced and developed by Lascaux Graphics. This software will allow students to generate graphs of complex variables in a four-dimensional space without requiring user programming. These graphs can be easily rotated in real time, zoomed, and scaled to permit close and varied examination. Exercises in the text that can be enhanced by the use of this program are denoted with an asterisk (*). The software is available for both PC and Macintosh platforms.

Preparation of this revision has been influenced by suggestions from a number of people. Specifically, there has been considerable input from the following reviewers: Harry Hochstadt, Polytechnic University; Meyer Jerison, Purdue University; Fred Rispoli, Dowling College; and Calvin Wilcox, University of Utah.

Constant interest and support have also been provided by Jacqueline R. Brown, Margret H. Höft, Michael A. Lachance, Ronald P. Morash, Frank J. Papp, Richard L. Patterson, and Gene G. Rae, as well as Jack Shira, Maggie Lanzillo, and James W. Bradley of the editorial staff at McGraw-Hill.

James Ward Brown

COMPLEX VARIABLES AND APPLICATIONS

CONTENTS

CHAPTER 1

COMPLEX NUMBERS

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane,* with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the *real axis,* it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers.* The y axis is, then, referred to as the *imaginary axis.*

It is customary to denote a complex number (x, y) by z, so that

$$z = (x, y). \tag{1}$$

The real numbers x and y are, moreover, known as the *real and imaginary parts* of z, respectively; and we write

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y. \tag{2}$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts. Thus $z_1 = z_2$ if and only if z_1 and z_2 correspond to the same point in the complex, or z, plane.

The *sum* $z_1 + z_2$ and the *product* z_1z_2 of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

(3) $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

(4) $$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$
$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and, if we think of a real number as either x or $(x, 0)$ and *let i denote the pure imaginary number* $(0, 1)$, it is clear that*

(5) $$z = x + iy.$$

Also, with the convention $z^2 = zz$, $z^3 = zz^2$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

(6) $$i^2 = -1.$$

In view of expression (5), definitions (3) and (4) become

(7) $$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

(8) $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs.

2. ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

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*In electrical engineering, the letter j is used instead of i.