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Michael Meyer

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PREFACE

The current, prolonged boom in the US and European stock markets has increased interest in the mathematics of security markets most notably the theory of stochastic integration. Existing books on the subject seem to belong to one of two classes. On the one hand there are rigorous accounts which develop the theory to great depth without particular interest in finance and which make great demands on the prerequisite knowledge and mathematical maturity of the reader. On the other hand treatments which are aimed at application to finance are often of a nontechnical nature providing the reader with little more than an ability to manipulate symbols to which no meaning can be attached. The present book gives a rigorous development of the theory of stochastic integration as it applies to the valuation of derivative securities. It is hoped that a satisfactory balance between aesthetic appeal, degree of generality, depth and ease of reading is achieved

Prerequisites are minimal. For the most part a basic knowledge of measure theoretic probability and Hilbert space theory is sufficient. Slightly more advanced functional analysis (Banach Alaoglu theorem) is used only once. The development begins with the theory of discrete time martingales, in itself a charming subject. From these humble origins we develop all the necessary tools to construct the stochastic integral with respect to a general continuous semimartingale. The limitation to continuous integrators greatly simplifies the exposition while still providing a reasonable degree of generality. A leisurely pace is assumed throughout, proofs are presented in complete detail and a certain amount of redundancy is maintained in the writing, all with a view to make the reading as effortless and enjoyable as possible.

The book is split into four chapters numbered I, II, III, IV. Each chapter has sections 1,2,3 etc. and each section subsections a,b,c etc. Items within subsections are numbered 1,2,3 etc. again. Thus III.4.a.2 refers to item 2 in subsection a of section 4 of Chapter III. However from within Chapter III this item would be referred to as 4.a.2. Displayed equations are numbered (0), (1), (2) etc. Thus II.3.b.eq.(5) refers to equation (5) of subsection b of section 3 of Chapter II. This same equation would be referred to as 3.b.eq.(5) from within Chapter II and as (5) from within the subsection wherein it occurs.

Very little is new or original and much of the material is standard and can be found in many books. The following sources have been used:
[Ca,Cb] I.5.b.1, I.5.b.2, I.7.b.0, I.7.b.1;
[CRS] I.2.b, I.4.a.2, I.4.b.0;
[CW] III.2.e.0, III.3.e.1, III.2.e.3;

[DD] II.1.a.6, II.2.a.1, II.2.a.2;
[DF] IV.3.e;
[DT] I.8.a.6, II.2.e.7, II.2.e.9, III.4.b.3, III.5.b.2;
[J] III.3.c.4, IV.3.c.3, IV.3.c.4, IV.3.d, IV.5.e, IV.5.h;
[K] II.1.a, II.1.b;
[KS] I.9.d, III.4.c.5, III.4.d.0, III.5.a.3, III.5.c.4, III.5.f.1, IV.1.c.3;
[MR] IV.4.d.0, IV.5.g, IV.5.j;
[RY] I.9.b, I.9.c, III.2.a.2, III.2.d.5.

To
my mother

SUMMARY OF NOTATION

Sets and numbers. $\mathbb{N}$ denotes the set of natural numbers ($\mathbb{N} = \{1, 2, 3, \dots\}$), R the set of real numbers, $R_+ = [0, +\infty)$, $\overline{R} = [-\infty, +\infty]$ the extended real line and R^n Euclidean n-space. $\mathcal{B}(R)$, $\mathcal{B}(\overline{R})$ and $\mathcal{B}(R^n)$ denote the Borel σ-field on R, $\overline{R}$ and R^n respectively. $\mathcal{B}$ denotes the Borel σ-field on R_+. For $a, b \in \overline{R}$ set $a \vee b = max\{a, b\}$, $a \wedge b = min\{a, b\}$, $a^+ = a \vee 0$ and $a^- = -a \wedge 0$.

$\Pi = [0, +\infty) \times \Omega$	domain of a stochastic process
$\mathcal{P}_g$	the progressive σ-field on Π (III.1.a).
$\mathcal{P}$	the predictable σ-field on Π (III.1.a).
$[\![S, T]\!] = \{\, (t, \omega) \mid S(\omega) \le t \le T(\omega) \,\}$ . . .	stochastic interval.

Random variables. $(\Omega, \mathcal{F}, P)$ the underlying probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-field. For a random variable X set $X^+ = X \vee 0 = 1_{[X>0]}X$ and $X^- = -X \wedge 0 = -1_{[X<0]}X = (-X)^+$. Let $\mathcal{E}(P)$ denote the set of all random variables X such that the expected value $E_P(X) = E(X) = E(X^+) - E(X^-)$ is defined ($E(X^+) < \infty$ or $E(X^-) < \infty$). For $X \in \mathcal{E}(P)$, $E_{\mathcal{G}}(X) = E(X|\mathcal{G})$ is the unique $\mathcal{G}$-measurable random variable Z in $\mathcal{E}(P)$ satisfying $E(1_G X) = E(1_G Z)$ for all sets $G \in \mathcal{G}$ (the conditional expectation of X with respect to $\mathcal{G}$).

Processes. Let $X = (X_t)_{t \ge 0}$ be a stochastic process and $T : \Omega \to [0, \infty]$ an optional time. Then X_T denotes the random variable $(X_T)(\omega) = X_{T(\omega)}(\omega)$ (sample of X along T, I.3.b, I.7.a). X^T denotes the process $X^T_t = X_{t \wedge T}$ (process X stopped at time T). $\mathcal{S}$, $\mathcal{S}_+$ and $\mathcal{S}^n$ denote the space of continuous semimartingales, continuous positive semimartingales and continuous R^n-valued semimartingales respectively. Let $X, Y \in \mathcal{S}$, $t \ge 0$, $\Delta = \{\, 0 = t_0 < t_1 < \dots, t_n = t \,\}$ a partition of the interval $[0, t]$ and set $\Delta_j X = X_{t_j} - X_{t_{j-1}}$, $\Delta_j Y = Y_{t_j} - Y_{t_{j-1}}$ and $\|\Delta\| = max_j(t_j - t_{j-1})$.

$Q_\Delta(X) = \sum(\Delta_j X)^2$	I.9.b, I.10.a, I.11.b.
$Q_\Delta(X, Y) = \sum \Delta_j X \Delta_j Y$.	I.10.a.
$\langle X, Y \rangle$	covariation process of X, Y (I.10.a, I.11.b). $\langle X, Y \rangle_t = \lim_{\|\Delta\| \to 0} Q_\Delta(X, Y)$ (limit in probability).
$\langle X \rangle = \langle X, X \rangle$	quadratic variation process of X (I.9.b).
u_X	(additive) compensator of X (I.11.a).
U_X	multiplicative compensator of $X \in \mathcal{S}_+$ (III.3.f).
$\mathbf{H}^2$	space of continuous, L^2-bounded martingales M with norm $\|M\|_{\mathbf{H}^2} = \sup_{t \ge 0} \|M_t\|_{L^2(P)}$ (I.9.a).

$\mathbf{H}^2_0 = \{\, M \in \mathbf{H}^2 \mid M_0 = 0 \,\}$.

Multinormal distribution and Brownian motion.

W	Brownian motion starting at zero.
$\mathcal{F}^W_t$	Augmented filtration generated by W (II.2.f).
$N(m, C)$	Normal distribution with mean $m \in R^k$ and covariance matrix C (II.1.a).
$N(d) = P(X \le d)$	X a standard normal variable in R^1.
$n_k(x) = (2\pi)^{-k/2} exp(-\|x\|^2/2)$ . .	Standard normal density in R^k (II.1.a).

Stochastic integrals, spaces of integrands. $H \bullet X$ denotes the integral process $(H \bullet X)_t = \int_0^t H_s \cdot dX_s$ and is defined for $X \in \mathcal{S}^n$ and $H \in L(X)$. $L(X)$ is the space of X-integrable processes H. If X is a continuous local martingale, $L(X) = L^2_{loc}(X)$ and in this case we have the subspaces $L^2(X) \subseteq \Lambda^2(X) \subseteq L^2_{loc}(X) = L(X)$. The integral processes $H \bullet X$ and associated spaces of integrands H are introduced step by step for increasingly more general integrators X:

Scalar valued integrators. Let M be a continuous local martingale. Then

μ_M . . . . . Doleans measure on $(\Pi, \mathcal{B} \times \mathcal{F})$ associated with M (III.2.a) $\mu_M(\Delta) = E_P\left[\int_0^\infty 1_\Delta(s,\omega)d\langle M\rangle_s(\omega)\right]$, $\Delta \in \mathcal{B} \times \mathcal{F}$.

$L^2(M)$ . . . . space $L^2(\Pi, \mathcal{P}_g, \mu_M)$ of all progressively measurable processes H satisfying $\|H\|^2_{L^2(M)} = E_P\left[\int_0^\infty H_s^2 d\langle M\rangle_s\right] < \infty$.

For $H \in L^2(M)$, $H \bullet M$ is the unique martingale in $\mathbf{H}^2_0$ satisfying $\langle H \bullet M, N\rangle = H \bullet \langle M, N\rangle$, for all continuous local martingales N (III.2.a.2). The spaces $\Lambda^2(M)$ and $L(M) = L^2_{loc}(M)$ of M-integrable processes H are then defined as follows:

$\Lambda^2(M)$ . . . . space of all progressively measurable processes H satisfying $1_{[0,t]}H \in L^2(M)$, for all $0 < t < \infty$.

$L(M) = L^2_{loc}(M)$ . . space of all progressively measurable processes H satisfying $1_{[\![0,T_n]\!]}H \in L^2(M)$, for some sequence (T_n) of optional times increasing to infinity, equivalently $\int_0^t H_s^2 d\langle M\rangle_s < \infty$, P-as., for all $0 < t < \infty$ (III.2.b).

If $H \in L^2(M)$, then $H \bullet M$ is a martingale in $\mathbf{H}^2$. If $H \in \Lambda^2(M)$, then $H \bullet M$ is a square integrable martingale (III.2.c.3).

Let now A be a continuous process with paths which are almost surely of bounded variation on finite intervals. For $\omega \in \Omega$, $dA_s(\omega)$ denotes the (signed) Lebesgue-Stieltjes measure on finite subintervals of $[0, +\infty)$ corresponding to the bounded variation function $s \mapsto A_s(\omega)$ and $|dA_s|(\omega)$ the associated total variation measure.

$L^1(A)$ . . . . . the space of all progressively measurable processes H such that $\int_0^\infty |H_s(\omega)|\, |dA_s|(\omega) < \infty$, for P-ae. $\omega \in \Omega$.

$L^1_{loc}(A)$ . . . . the space of all progressively measurable processes H such that $1_{[0,t]}H \in L^1(A)$, for all $0 < t < \infty$.

For $H \in L^1_{loc}(A)$ the integral process $I_t = (H \bullet A)_t = \int_0^t H_s dA_s$ is defined pathwise as $I_t(\omega) = \int_0^t H_s(\omega)dA_s(\omega)$, for P-ae. $\omega \in \Omega$.

Assume now that X is a continuous semimartingale with semimartingale decomposition $X = A + M$ ($A = u_X$, M a continuous local martingale, I.11.a). Then $L(X) = L^1_{loc}(A) \cap L^2_{loc}(M)$. Thus $L(X) = L^2_{loc}(X)$, if X is a local martingale.
For $H \in L(X)$ set $H \bullet X = H \bullet A + H \bullet M$. Then $H \bullet X$ is the unique continuous semimartingale satisfying $(H \bullet X)_0 = 0$, $u_{H \bullet X} = H \bullet u_X$ and $\langle H \bullet X, Y\rangle = H \bullet \langle X, Y\rangle$, for all $Y \in \mathcal{S}$ (III.4.a.2). In particular $\langle H \bullet X\rangle = \langle H \bullet X, H \bullet X\rangle = H^2 \bullet \langle X\rangle$. In

other words $\langle H\bullet X\rangle_t = \int_0^t H_s^2 d\langle X\rangle_s$. If the integrand H is continuous we have the representation

$$\int_0^t H_s dX_s = \lim_{\|\Delta\|\to 0} S_\Delta(H,X)$$

(limit in probability), where $S_\Delta(H,X) = \sum H_{t_{j-1}}(X_{t_j} - X_{t_{j-1}})$ for Δ as above (III.2.e.0). The (deterministic) process $\mathbf{t}$ defined by $\mathbf{t}(t) = t$, $t \geq 0$, is a continuous semimartingale, in fact a bounded variation process. Thus the spaces $L(\mathbf{t})$ and $L^1_{loc}(\mathbf{t})$ are defined and in fact $L(\mathbf{t}) = L^1_{loc}(\mathbf{t})$.

Vector valued integrators. Let $X \in \mathcal{S}^d$ and write $X = (X^1, X^2, \dots, X^d)'$ (column vector), with $X^j \in \mathcal{S}$. Then $L(X)$ is the space of all R^d-valued processes $H = (H^1, H^2, \dots, H^d)'$ such that $H^j \in L(X^j)$, for all $j = 1, 2, \dots, d$. For $H \in L(X)$,

$$H\bullet X = \textstyle\sum_j H^j \bullet X^j, \quad (H\bullet X)_t = \int_0^t H_s\cdot dX_s = \sum_j \int_0^t H_s^j dX_s^j,$$
$$dX = (dX^1, dX^2, \dots, dX^d)', \quad H_s\cdot dX_s = \textstyle\sum_j H_s^j dX_s^j.$$

If X is a continuous local martingale (all the X^j continuous local martingales), the spaces $L^2(X)$, $\Lambda^2(X)$ are defined analogously. If $H \in \Lambda^2(X)$, then $H\bullet X$ is a square integrable martingale; if $H \in L^2(X)$, then $H\bullet X \in \mathbf{H}^2$ (III.2.c.3, III.2.f.3).

In particular, if W is an R^d-valued Brownian motion, then

$L^2(W)$ . . . . space of all progressively measurable processes H such that $\|H\|^2_{L^2(W)} = E_P \int_0^\infty \|H_s\|^2 ds < \infty$.

$\Lambda^2(W)$ . . . . . . space of all progressively measurable processes H such that $1_{[0,t]}H \in L^2(W)$, for all $0 < t < \infty$.

$L(W) = L^2_{loc}(W)$ . . space of all progressively measurable processes H such that $\int_0^t \|H_s\|^2 ds < \infty$, P-as., for all $0 < t < \infty$.

If $H \in L^2(W)$, then $H\bullet W$ is a martingale in $\mathbf{H}^2$ with $\|H\bullet W\|_{\mathbf{H}^2} = \|H\|_{L^2(W)}$. If $H \in \Lambda^2(W)$, then $H\bullet W$ is a square integrable martingale (III.2.f.3, III.2.f.5).

Stochastic differentials. If $X \in \mathcal{S}^n$, $Z \in \mathcal{S}$ write $dZ = H\cdot dX$ if $H \in L(X)$ and $Z = Z_0 + H\bullet X$, that is, $Z_t = Z_0 + \int_0^t H_s\cdot dX_s$, for all $t \geq 0$. Thus $d(H\bullet X) = H\cdot dX$. We have $dZ = dX$ if and only if $Z - X$ is constant (in time). Likewise $KdZ = HdX$ if and only if $K \in L(Z)$, $H \in L(X)$ and $K\bullet Z = H\bullet X$ (III.3.b). With the process $\mathbf{t}$ as above we have $d\mathbf{t}(t) = dt$.

Local martingale exponential. Let M be a continuous, real valued local martingale. Then the local martingale exponential $\mathcal{E}(M)$ is the process

$$X_t = \mathcal{E}_t(M) = exp\big(M_t - \tfrac{1}{2}\langle M\rangle_t\big).$$

$X = \mathcal{E}(M)$ is the unique solution to the exponential equation $dX_t = X_t dM_t$, $X_0 = 1$. If $\gamma \in L(M)$, then all solutions X to the equation $dX_t = \gamma_t X_t dM_t$ are given by $X_t = X_0\mathcal{E}_t(\gamma\bullet M)$. If W is an R^d-valued Brownian motion and $\gamma \in L(W)$, then all solutions to the equation $dX_t = \gamma_t X_t \cdot dW_t$ are given by

$$X_t = X_0\mathcal{E}_t(\gamma\bullet W) = X_0 exp\big(-\tfrac{1}{2}\int_0^t \|\gamma_s\|^2 ds + \int_0^t \gamma_s \cdot dW_s\big) \quad \text{(III.4.b)}.$$

Finance. Let B be a market (IV.3.b), $Z \in \mathcal{S}$ and $A \in \mathcal{S}_+$.

$Z_t^A = Z_t/A_t$ . . . Z expressed in A-numeraire units.
$B(t,T)$ . . . . . Price at time t of the zero coupon bond maturing at time T.
$B_0(t)$ . . . . . . Riskless bond.
P_A . . . . A-numeraire measure (IV.3.d).
P_T . . . . Forward martingale measure at date T (IV.3.f).
W_t^T . . . . Process which is a Brownian motion with respect to P_T.
$L(t,T_j)$ . . . . . Forward Libor set at time T_j for the accrual interval $[T_j, T_{j+1}]$.
$L(t)$ . . . . Process $\big(L(t,T_0),\dots,L(t,T_{n-1})\big)$ of forward Libor rates.

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CHAPTER I

Martingale Theory

Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\overline{R} = [-\infty, +\infty]$ denote the extended real line and $\mathcal{B}(\overline{R})$ and $\mathcal{B}(R^n)$ the Borel σ-fields on $\overline{R}$ and R^n respectively.

A *random object* on $(\Omega, \mathcal{F}, P)$ is a measurable map $X : (\Omega, \mathcal{F}, P) \to (\Omega_1, \mathcal{F}_1)$ with values in some measurable space $(\Omega_1, \mathcal{F}_1)$. P_X denotes the distribution of X (appendix B.5). If Q is any probability on $(\Omega_1, \mathcal{F}_1)$ we write $X \sim Q$ to indicate that $P_X = Q$. If $(\Omega_1, \mathcal{F}_1) = (R^n, \mathcal{B}(R^n))$ respectively $(\Omega_1, \mathcal{F}_1) = (\overline{R}, \mathcal{B}(\overline{R}))$, X is called a *random vector* respectively *random variable*. In particular random variables are extended real valued.

For extended real numbers a, b we write $a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$. If X is a random variable, the set $\{\, \omega \in \Omega \mid X \geq 0 \,\}$ will be written as $[X \geq 0]$ and its probability denoted $P([X \geq 0])$ or, more simply, $P(X \geq 0)$. We set $X^+ = X \vee 0 = 1_{[X>0]}X$ and $X^- = (-X)^+$. Thus $X^+, X^- \geq 0$, $X^+X^- = 0$ and $X = X^+ - X^-$.

For nonnegative X let $E(X) = \int_\Omega X dP$ and let $\mathcal{E}(P)$ denote the family of all random variables X such that at least one of $E(X^+)$, $E(X^-)$ is finite. For $X \in \mathcal{E}(P)$ set $E(X) = E(X^+) - E(X^-)$ (*expected value* of X). This quantity will also be denoted $E_P(X)$ if dependence on the probability measure P is to be made explicit.

If $X \in \mathcal{E}(P)$ and $A \in \mathcal{F}$ then $1_A X \in \mathcal{E}(P)$ and we write $E(X; A) = E(1_A X)$. The expression "P-almost surely" will be abbreviated "P-as.". Since random variables X, Y are extended real valued, the sum $X + Y$ is not defined in general. However it is defined (P-as.) if both $E(X^+)$ and $E(Y^+)$ are finite, since then $X, Y < +\infty$, P-as., or both $E(X^-)$ and $E(Y^-)$ are finite, since then $X, Y > -\infty$, P-as.

An *event* is a set $A \in \mathcal{F}$, that is, a measurable subset of Ω. If (A_n) is a sequence of events let $[A_n \, i.o.] = \bigcap_m \bigcup_{n \geq m} A_n = \{\, \omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n \,\}$.

Borel Cantelli Lemma. *(a) If $\sum_n P(A_n) < \infty$ then $P(A_n \, i.o.) = 0$.*
(b) If the events A_n are independent and $\sum_n P(A_n) = \infty$ then $P(A_n \, i.o.) = 1$.
(c) If $P(A_n) \geq \delta$, for all $n \geq 1$, then $P(A_n i.o.) \geq \delta$.

Proof. (a) Let $m \geq 1$. Then $0 \leq P(A_n \, i.o.) \leq \sum_{n \geq m} P(A_n) \to 0$, as $m \uparrow \infty$.
(b) Set $A = [A_n \, i.o.]$. Then $P(A^c) = \lim_m P\big(\bigcap_{n \geq m} A_n^c\big) = \lim_m \prod_{n \geq m} P(A_n^c) = \lim_m \prod_{n \geq m} (1 - P(A_n)) = 0$. (c) Since $P(A_n i.o.) = \lim_m P\big(\bigcup_{n \geq m} A_n\big)$. ∎