

# **SEMI-RIEMANNIAN GEOMETRY**

**With Applications to Relativity**

***BARRETT O'NEILL***

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WITH APPLICATIONS TO RELATIVITY

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# **SEMI-RIEMANNIAN GEOMETRY**

**WITH APPLICATIONS TO RELATIVITY**

**This is a volume in  
PURE AND APPLIED MATHEMATICS  
A Series of Monographs and Textbooks**

**Editors: SAMUEL EILENBERG AND HYMAN BASS**

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## PREFACE

This book is an exposition of *semi-Riemannian geometry* (also called *pseudo-Riemannian geometry*)—the study of a smooth manifold furnished with a metric tensor of arbitrary signature. The principal special cases are Riemannian geometry, where the metric is positive definite, and Lorentz geometry. For many years these two geometries have developed almost independently: Riemannian geometry reformulated in coordinate-free fashion and directed toward global problems, Lorentz geometry in classical tensor notation devoted to general relativity. More recently, this divergence has been reversed as physicists, turning increasingly toward invariant methods, have produced results of compelling mathematical interest.

After establishing the requisite language of manifolds and tensors (Chapters 1 and 2), the plan of the book is to develop the foundations of semi-Riemannian geometry in the simplest way and without regard to signature, allowing the Riemannian and Lorentz cases to appear as needed (Chapters 3–5 and 7). Then in the latter half of the book two threads are followed. One uses the notion of isometry to develop algebraic aspects of semi-Riemannian geometry: manifolds of constant curvature, symmetric spaces, and homogeneous spaces (Chapters 8, 9, and 11); the introductions to these chapters will give a more detailed description of their contents. The other thread applies Lorentz geometry to special and general relativity (Chapters 6, 12, and 13). The fact that relativity theory is expressed in terms of Lorentz geometry is lucky for geometers, who can thus penetrate surprisingly quickly into cosmology (redshift, expanding universe, and big bang) and, a topic no less interesting geometrically, the gravitation of a single star (perihelion precession, bending of

light, and black holes). The tendency of the spacetimes in Chapters 12 and 13 to have singularities (big bang and black holes) is accounted for in abstract Lorentz terms by two theorems, due respectively to S. W. Hawking and R. Penrose; these are the goals of Chapter 14.

The general approach of the book is coordinate-free; however, coordinates are not neglected. Typically, geometric objects are defined invariantly and then described in terms of coordinates. In particular, the definition of a tensor I have adopted converts almost automatically into the classical coordinate formulation. A number of key proofs are given in classical notation. This attitude is only reasonable in view of the vast literature in each style.

The basic prerequisites for the book are modest: a good working knowledge of multivariable differential calculus, a firm belief in the existence and uniqueness theorems of ordinary differential equations, and an acquaintance with the fundamentals of point set topology and algebra. Later on, a knowledge of fundamental groups, covering spaces, and Lie groups is required; the necessary background in these topics is outlined briefly in Appendixes A and B. A college course in physics (particularly Newtonian mechanics) is required, not to read this book, but to appreciate the transformation and unification of Newtonian concepts effected by Einstein's relativistic geometry and the remarkable way the old and new theories—so different at base—reach approximate agreement on, say, the running of the solar system (Appendix C versus Chapter 13).

In the early chapters (1–5 and 7) the logical ordering is fairly strict. Thereafter the two branches—8,9,11 and 6,12,13—are almost independent. (Chapters 12 and 13 require only an occasional reference to Chapters 9 and perhaps 8.) Chapter 10 is used in Chapters 11 and 14. Otherwise Chapter 14, though strongly motivated by Chapters 12 and 13, depends logically on only the early chapters.

Following each chapter are a number of exercises; these are meant to be workable without undue strain. In each chapter a single sequence of numbers designates collectively the theorems, lemmas, examples, and so on. For instance, Lemma 5.12 is the twelfth designated item in Chapter 5, not the twelfth lemma. Within a given chapter, the chapter number is omitted. Initials in square brackets, e.g., [SW], direct the reader to the References.

It is a pleasure to express my gratitude to the authors of the following brilliant and very different books: S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*; C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*; R. K. Sachs and H. Wu, *General Relativity for Mathematicians*.

## NOTATION AND TERMINOLOGY

The following notations are among the most frequently used throughout the book:

$M, N$	manifolds	$p, q$	points
$f, g, h$	real-valued functions	$\alpha, \beta, \gamma$	curves
$v, w$	vectors	$V, W, X, Y$	vector fields
$\phi, \psi$	mappings	$\mathcal{U}, \mathcal{V}$	open sets
$\xi = (x^1, \dots, x^n)$		coordinate system	

$\mathbf{R}$  is the real number field,  $I$  denotes an open interval in  $\mathbf{R}$ , and, for example,  $[a, b] = \{r \in \mathbf{R}: a \leq r < b\}$ . The identity map is  $\text{id}$ ;  $\phi \circ \psi$  is the composite mapping that sends  $p$  to  $\phi(\psi p)$ . See Appendix B for Lie group notation such as  $GL(n, \mathbf{R})$ .

A mapping  $\phi: M \rightarrow N$  is *one-to-one* (injective) if  $p \neq q$  implies  $\phi p \neq \phi q$ . The *image* of  $\phi$  is  $\{\phi p: p \in M\} \subset N$ , and  $\phi$  is *onto* (surjective) if  $\text{image } \phi = N$ . (Inclusion  $B \subset N$  does not exclude equality  $B = N$ .) If  $B \subset N$  then  $\phi^{-1}(B) = \{p \in M: \phi p \in B\}$ , and when  $\phi$  is one-to-one and onto,  $\phi^{-1}$  also denotes the inverse mapping of  $\phi$ .

If  $\pi \circ \tilde{\phi} = \phi$ , then  $\tilde{\phi}$  is called a *lift* of  $\phi$  through  $\pi$ . A lift of the identity map is called a *cross section* (or merely a *section*).

A *linear isomorphism* of vector spaces is a linear transformation that is one-to-one and onto, hence is *invertible*.

A subset  $A$  of a topological space has closure  $\bar{A}$ , interior  $\text{int } A$ , and boundary  $\text{bd } A$ .



# CONTENTS

<i>Preface</i>	xi
<i>Notation and Terminology</i>	xiii

## 1. MANIFOLD THEORY

Smooth Manifolds	1
Smooth Mappings	4
Tangent Vectors	6
Differential Maps	9
Curves	10
Vector Fields	12
One-Forms	14
Submanifolds	15
Immersions and Submersions	19
Topology of Manifolds	21
Some Special Manifolds	24
Integral Curves	27

## 2. TENSORS

Basic Algebra	34
Tensor Fields	35
Interpretations	36
Tensors at a Point	37
Tensor Components	39
Contraction	40
Covariant Tensors	42
Tensor Derivations	43

## **vi Contents**

Symmetric Bilinear Forms	46
Scalar Products	47

### **3. SEMI-RIEMANNIAN MANIFOLDS**

Isometries	58
The Levi-Civita Connection	59
Parallel Translation	65
Geodesics	67
The Exponential Map	70
Curvature	74
Sectional Curvature	77
Semi-Riemannian Surfaces	80
Type-Changing and Metric Contraction	81
Frame Fields	84
Some Differential Operators	85
Ricci and Scalar Curvature	87
Semi-Riemannian Product Manifolds	89
Local Isometries	90
Levels of Structure	93

### **4. SEMI-RIEMANNIAN SUBMANIFOLDS**

Tangents and Normals	97
The Induced Connection	98
Geodesics in Submanifolds	102
Totally Geodesic Submanifolds	104
Semi-Riemannian Hypersurfaces	106
Hyperquadrics	108
The Codazzi Equation	114
Totally Umbilic Hypersurfaces	116
The Normal Connection	118
A Congruence Theorem	120
Isometric Immersions	121
Two-Parameter Maps	122

### **5. RIEMANNIAN AND LORENTZ GEOMETRY**

The Gauss Lemma	126
Convex Open Sets	129
Arc Length	131
Riemannian Distance	132
Riemannian Completeness	138
Lorentz Causal Character	140
Timecones	143
Local Lorentz Geometry	146
Geodesics in Hyperquadrics	149

Geodesics in Surfaces	150
Completeness and Extendibility	154

## 6. SPECIAL RELATIVITY

Newtonian Space and Time	158
Newtonian Space-Time	160
Minkowski Spacetime	163
Minkowski Geometry	164
Particles Observed	167
Some Relativistic Effects	171
Lorentz-Fitzgerald Contraction	174
Energy-Momentum	176
Collisions	179
An Accelerating Observer	181

## 7. CONSTRUCTIONS

Deck Transformations	185
Orbit Manifolds	187
Orientability	189
Semi-Riemannian Coverings	191
Lorentz Time-Orientability	194
Volume Elements	194
Vector Bundles	197
Local Isometries	200
Matched Coverings	203
Warped Products	204
Warped Product Geodesics	207
Curvature of Warped Products	209
Semi-Riemannian Submersions	212

## 8. SYMMETRY AND CONSTANT CURVATURE

Jacobi Fields	215
Tidal Forces	218
Locally Symmetric Manifolds	219
Isometries of Normal Neighborhoods	221
Symmetric Spaces	224
Simply Connected Space Forms	227
Transvections	231

## 9. ISOMETRIES

Semiorthogonal Groups	233
Some Isometry Groups	239

## **viii Contents**

Time-Orientability and Space-Orientability	240
Linear Algebra	242
Space Forms	243
Killing Vector Fields	249
The Lie Algebra $\mathfrak{i}(M)$	252
$I(M)$ as Lie Group	254
Homogeneous Spaces	257

## **10. CALCULUS OF VARIATIONS**

First Variation	263
Second Variation	266
The Index Form	268
Conjugate Points	270
Local Minima and Maxima	272
Some Global Consequences	277
The Endmanifold Case	280
Focal Points	281
Applications	286
Variation of $E$	288
Focal Points along Null Geodesics	290
A Causality Theorem	293

## **11. HOMOGENEOUS AND SYMMETRIC SPACES**

More about Lie Groups	300
Bi-Invariant Metrics	304
Coset Manifolds	306
Reductive Homogeneous Spaces	310
Symmetric Spaces	315
Riemannian Symmetric Spaces	319
Duality	321
Some Complex Geometry	323

## **12. GENERAL RELATIVITY; COSMOLOGY**

Foundations	332
The Einstein Equation	336
Perfect Fluids	337
Robertson-Walker Spacetimes	341
The Robertson-Walker Flow	345
Robertson-Walker Cosmology	347
Friedmann Models	350
Geodesics and Redshift	353
Observer Fields	358
Static Spacetimes	360

### 13. SCHWARZSCHILD GEOMETRY

Building the Model	364
Geometry of $N$ and $B$	368
Schwarzschild Observers	371
Schwarzschild Geodesics	372
Free Fall Orbits	374
Perihelion Advance	378
Lightlike Orbits	380
Stellar Collapse	384
The Kruskal Plane	386
Kruskal Spacetime	389
Black Holes	392
Kruskal Geodesics	395

### 14. CAUSALITY IN LORENTZ MANIFOLDS

Causality Relations	402
Quasi-Limits	404
Causality Conditions	407
Time Separation	409
Achronal Sets	413
Cauchy Hypersurfaces	415
Warped Products	417
Cauchy Developments	419
Spacelike Hypersurfaces	425
Cauchy Horizons	428
Hawking's Singularity Theorem	431
Penrose's Singularity Theorem	434

### APPENDIX A. FUNDAMENTAL GROUPS AND COVERING MANIFOLDS

441

### APPENDIX B. LIE GROUPS

Lie Algebras	447
Lie Exponential Map	449
The Classical Groups	450

### APPENDIX C. NEWTONIAN GRAVITATION

453

<i>References</i>	456
-------------------	-----

<i>Index</i>	459
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# 1 MANIFOLD THEORY

Generally speaking, a manifold is a topological space that locally resembles Euclidean space. A smooth manifold is a manifold  $M$  for which this resemblance is sharp enough to permit the establishment of partial differentiation—in fact, all the essential features of calculus—on  $M$ . Smooth manifolds are thus the natural setting for “calculus in the large.”

## SMOOTH MANIFOLDS

*Euclidean  $n$ -space  $\mathbf{R}^n$*  is the set of all  $n$ -tuples  $p = (p_1, \dots, p_n)$  of real numbers. We assume in particular a familiarity with its structure as a vector space and as a topological space. The natural inner product of  $\mathbf{R}^n$  is the *dot product*  $p \cdot q = \sum p_i q_i$ , with *norm*  $|p| = \sqrt{p \cdot p}$ . The resulting *metric*  $d(p, q) = |p - q|$  is compatible with the topology of  $\mathbf{R}^n$ .

A real-valued function  $f$  defined on an open set  $\mathcal{U}$  of  $\mathbf{R}^n$  is *smooth* (or equivalently,  $C^\infty$ ) provided all mixed partial derivatives of  $f$ —of all orders—exist and are continuous at every point of  $\mathcal{U}$ .

For  $1 \leq i \leq n$ , let  $u^i: \mathbf{R}^n \rightarrow \mathbf{R}$  be the function that sends each point  $p = (p_1, \dots, p_n)$  to its  $i$ th coordinate  $p_i$ . Then  $u^1, \dots, u^n$  are the *natural coordinate functions* of  $\mathbf{R}^n$ .

A function  $\phi$  from an open set  $\mathcal{U}$  of  $\mathbf{R}^m$  to  $\mathbf{R}^n$  is *smooth* provided each real-valued function  $u^i \circ \phi$  is smooth ( $1 \leq i \leq n$ ).

We can now make precise the resemblance to Euclidean space mentioned above. A *coordinate system* (or *chart*) in a topological space  $S$  is a homeomorphism  $\xi$  of an open set  $\mathcal{U}$  of  $S$  onto an open set  $\xi(\mathcal{U})$  of  $\mathbf{R}^n$ . If we write

$$\xi(p) = (x^1(p), \dots, x^n(p)) \quad \text{for each } p \in \mathcal{U},$$

the resulting functions  $x^1, \dots, x^n$  are called the *coordinate functions* of  $\xi$ . Thus

$$\xi = (x^1, \dots, x^n): \mathcal{U} \rightarrow \mathbb{R}^n.$$

Here we call  $n$  the *dimension* of  $\xi$ . Note the identity  $u^i \circ \xi = x^i$ .

Two  $n$ -dimensional coordinate systems  $\xi$  and  $\eta$  in  $S$  *overlap smoothly* provided the functions  $\xi \circ \eta^{-1}$  and  $\eta \circ \xi^{-1}$  are both smooth. Explicitly, if  $\xi: \mathcal{U} \rightarrow \mathbb{R}^n$  and  $\eta: \mathcal{V} \rightarrow \mathbb{R}^n$ , then  $\eta \circ \xi^{-1}$  is defined on the open set  $\xi(\mathcal{U} \cap \mathcal{V})$  and carries it to  $\eta(\mathcal{U} \cap \mathcal{V})$ —while its inverse function  $\xi \circ \eta^{-1}$  runs in the opposite direction (see Figure 1). These functions are then required to be smooth in the usual Euclidean sense defined above. This condition is considered to hold trivially if  $\mathcal{U}$  and  $\mathcal{V}$  do not meet.

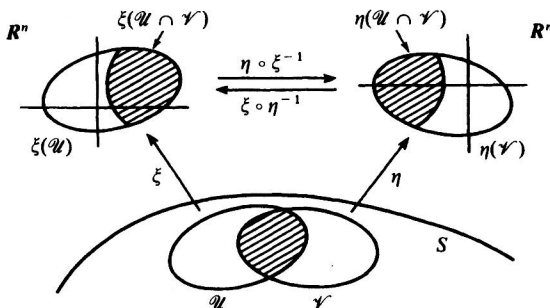


Figure 1.

**1. Definition.** An *atlas*  $\mathcal{A}$  of dimension  $n$  on a space  $S$  is a collection of  $n$ -dimensional coordinate systems in  $S$  such that

- (A1) each point of  $S$  is contained in the domain of some coordinate system in  $\mathcal{A}$ , and
- (A2) any two coordinate systems in  $\mathcal{A}$  overlap smoothly.

An atlas on  $S$  makes it possible to do calculus consistently on all of  $S$ . But different atlases may produce the same calculus, a technical difficulty eliminated as follows. Call an atlas  $\mathcal{C}$  on  $S$  *complete* if  $\mathcal{C}$  contains each coordinate system in  $S$  that overlaps smoothly with every coordinate system in  $\mathcal{C}$ .

**2. Lemma.** Each atlas  $\mathcal{A}$  on  $S$  is contained in a unique complete atlas.

*Proof.* If  $\mathcal{A}$  has dimension  $n$ , let  $\mathcal{A}'$  be the set of all  $n$ -dimensional coordinate systems in  $S$  that overlap smoothly with every one contained in  $\mathcal{A}$ .

- (a)  $\mathcal{A}'$  is an atlas (of the same dimension as  $\mathcal{A}$ ).

Since (A1) is obvious, consider (A2). If  $\eta_1, \eta_2 \in \mathcal{A}'$ , then by symmetry we need only prove that the function  $\eta_1 \circ \eta_2^{-1}$  is Euclidean smooth. For any point  $p \in \mathbb{R}^n$  in its domain, choose a  $\xi \in \mathcal{A}$  whose domain contains  $\eta_2^{-1}(p)$ . As a composition of smooth functions,  $(\eta_1 \circ \xi^{-1}) \circ (\xi \circ \eta_2^{-1})$  is smooth. Since this function equals  $\eta_1 \circ \eta_2^{-1}$  on a neighborhood of  $p$ , the latter is smooth on that neighborhood. Smoothness being a local property, (a) follows.

(b)  $\mathcal{A}'$  is complete. If a coordinate system  $\xi$  in  $S$  overlaps smoothly with every element of  $\mathcal{A}' \supset \mathcal{A}$ , then by definition  $\xi \in \mathcal{A}'$ .

(c)  $\mathcal{A}'$  is the unique complete atlas containing  $\mathcal{A}$ .

If  $\mathcal{C}$  is another, then since  $\mathcal{C}$  contains  $\mathcal{A}$ , (A2) guarantees that  $\mathcal{C} \subset \mathcal{A}'$ . But then (A2) implies  $\mathcal{A}' \subset \mathcal{C}$ . ■

**3. Definition.** A smooth manifold  $M$  is a Hausdorff space furnished with a complete atlas.

There are many variants of the notion of manifold but for us *manifold* will mean *smooth manifold* as above. Any atlas  $\mathcal{A}$  on a Hausdorff space makes it a manifold since we agree always to use the unique complete atlas containing  $\mathcal{A}$  to fulfill Definition 3. The *dimension*  $n = \dim M$  of a manifold  $M$  is the dimension of its atlas, and is often indicated by the notation  $M^n$ .

A coordinate system  $\xi$  in a manifold  $M$  is a coordinate system belonging to the complete atlas of  $M$ . If the domain  $\mathcal{U}$  of  $\xi$  contains the point  $p \in M$ , then  $\xi$  is called a coordinate system at  $p$  and  $\mathcal{U}$  a *coordinate neighborhood* of  $p$ .

If  $\xi$  is a coordinate system in  $M$  and  $\mathcal{V}$  is an open set contained in the domain of  $\xi$ , then by completeness  $\xi|_{\mathcal{V}}$  is also a coordinate system in  $M$ .

**4. Examples of Manifolds.** (1) The identity map  $(u^1, \dots, u^n)$  of  $\mathbb{R}^n$ , by itself, is an atlas. From now on,  $\mathbb{R}^n$  will denote the resulting  $n$ -dimensional manifold, called *Euclidean  $n$ -space*.

(2) *The sphere  $S^n$ .* Let  $S^n$  be the subspace  $\{a \in \mathbb{R}^{n+1} : |a| = 1\}$  of  $\mathbb{R}^{n+1}$ . For each  $1 \leq i \leq n+1$ , let  $\mathcal{U}_i, [\mathcal{U}_i^-]$  be the open hemisphere consisting of all points  $a$  with  $a_i > 0$  [ $a_i < 0$ ]. The restriction to  $\mathcal{U}_i$  or  $\mathcal{U}_i^-$  of the coordinate functions  $u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^{n+1}$  gives a coordinate system in the space  $S^n$ . It is easy to check that the  $2(n+1)$  coordinate systems gotten in this way constitute an atlas on  $S^n$  making it an  $n$ -dimensional manifold.

(3) A two-dimensional manifold is often called a *surface*, and generally speaking, the objects called surfaces in elementary calculus (torus, cylinder, paraboloid, etc.) are two-dimensional manifolds.

We now consider two simple ways to get new manifolds from old.

Let  $\mathcal{U}$  be an open set in a manifold  $M$ . Let  $\mathcal{A}'$  be the set of all coordinate systems  $\xi$  in  $M$  such that the domain of  $\xi$  is contained in  $\mathcal{U}$ . By the remark



preceding Example 4 these domains cover  $\mathcal{U}$ . Hence  $\mathcal{A}'$  is an atlas on  $\mathcal{U}$ , making it a manifold called an *open submanifold* of  $M$ . Open sets of a manifold will always be considered to be open submanifolds.

If  $M$  and  $N$  are manifolds, let

$$\xi = (x^1, \dots, x^m): \mathcal{U} \rightarrow \mathbf{R}^m \quad \text{and} \quad \eta = (y^1, \dots, y^n): \mathcal{V} \rightarrow \mathbf{R}^n$$

be coordinate systems in  $M$  and  $N$ , respectively. The product function  $\xi \times \eta: \mathcal{U} \times \mathcal{V} \rightarrow \mathbf{R}^{m+n}$  is defined by

$$(\xi \times \eta)(p, q) = (x^1(p), \dots, x^m(p), y^1(q), \dots, y^n(q)).$$

Evidently  $\xi \times \eta$  is a coordinate system in the Hausdorff space  $M \times N$ , and it is easy to see that any two such *product coordinate systems* in  $M \times N$  overlap smoothly.

**5. Lemma.** If  $M$  and  $N$  are manifolds, then the set of all product coordinate systems in  $M \times N$  is an atlas on  $M \times N$  making it the *product manifold* of  $M$  and  $N$ .

The dimension of  $M \times N$  is  $\dim M + \dim N$ . This construction extends in an obvious way to the product of any finite number of manifolds. Indeed Euclidean space  $\mathbf{R}^n$ , as in Example 4, is exactly the product manifold  $\mathbf{R}^1 \times \dots \times \mathbf{R}^1$  ( $n$  factors).

## SMOOTH MAPPINGS

Consider first the special case of a real-valued function  $f$  on a manifold  $M$ . If  $\xi: \mathcal{U} \rightarrow \mathbf{R}^n$  is a coordinate system in  $M$ , then the composite function  $f \circ \xi^{-1}: \xi(\mathcal{U}) \rightarrow \mathbf{R}^1$  is called the *coordinate expression* for  $f$  in terms of  $\xi$ . In fact,

$$f = (f \circ \xi^{-1})(x^1, \dots, x^n) \quad \text{on } \mathcal{U}.$$

(Compare, from elementary calculus, expressing a function  $f = f(x, y)$  in terms of polar coordinates.) It is natural then to define a function  $f: M \rightarrow \mathbf{R}$  to be *smooth* provided that for every coordinate system  $\xi$  in  $M$  the coordinate expression  $f \circ \xi^{-1}$  is smooth in the usual Euclidean sense. Let  $\mathfrak{F}(M)$  be the set of all smooth real-valued functions on  $M$ . If  $f$  and  $g$  are smooth functions on  $M$  so is their sum  $f + g$  and product  $fg$ . The usual algebraic rules hold for these two operations, making  $\mathfrak{F}(M)$  a commutative ring. Multiplicative inverses do not exist in general, but if  $f \in \mathfrak{F}(M)$  is never zero, then  $1/f \in \mathfrak{F}(M)$ .

The notion of smoothness extends from a real-valued function to an arbitrary mapping of manifolds using the same idea: that coordinate expressions must be Euclidean smooth.