SEMI-RIEMANNIAN GEOMETRY

With Applications to Relativity

BARRETT O'NEILL

SEMI-RIEMANNIAN GEOMETRY

WITH APPLICATIONS TO RELATIVITY

BARRETT O'NEILL

Department of Mathematics
University of California
Los Angeles, California

Academic Press
An Imprint of Elsevier
San Diego New York Boston
London Sydney Tokyo Toronto

Academic Press
An Imprint of Elsevier
525 B Street, Suite 1900, San Diego, California 92101-4495, USA
http://www.academicpress.com

Academic Press
32 Jamestown Road, London NW1 7BY, UK
http://www.academicpress.com

Library of Congress Cataloging-in-Publication

O'Neill, Barrett.
Semi-Riemannian geometry.

(Pure and Applied Mathematics ;)
Bibliography: p.
ISBN-13: 978-0-12-526740-3 ISBN-10: 0-12-526740-1
1. Geometry, Riemannian. 2. Manifolds (Mathematics) 3. Calculus of tensors.
4. Reletivity (Physics) I. Title. II. Series:
Pure and applied mathematics (Academic Press) ;
QA3.P8 [QA649] 510s [516.3'73] 82-13917

ISBN-13: 978-0-12-526740-3
ISBN-10: 0-12-526740-1

PRINTED IN THE UNITED STATES OF AMERICA
08 09 10 11 12 152 14 13

SEMI-RIEMANNIAN GEOMETRY

WITH APPLICATIONS TO RELATIVITY

This is a volume in
PURE AND APPLIED MATHEMATICS

A Series of Monographs and Textbooks

Editors: SAMUEL EILENBERG AND HYMAN BASS

A complete list of titles in this series is available from the Publisher upon request.

PREFACE

This book is an exposition of *semi-Riemannian geometry* (also called *pseudo-Riemannian geometry*)—the study of a smooth manifold furnished with a metric tensor of arbitrary signature. The principal special cases are Riemannian geometry, where the metric is positive definite, and Lorentz geometry. For many years these two geometries have developed almost independently: Riemannian geometry reformulated in coordinate-free fashion and directed toward global problems, Lorentz geometry in classical tensor notation devoted to general relativity. More recently, this divergence has been reversed as physicists, turning increasingly toward invariant methods, have produced results of compelling mathematical interest.

After establishing the requisite language of manifolds and tensors (Chapters 1 and 2), the plan of the book is to develop the foundations of semi-Riemannian geometry in the simplest way and without regard to signature, allowing the Riemannian and Lorentz cases to appear as needed (Chapters 3–5 and 7). Then in the latter half of the book two threads are followed. One uses the notion of isometry to develop algebraic aspects of semi-Riemannian geometry: manifolds of constant curvature, symmetric spaces, and homogeneous spaces (Chapters 8, 9, and 11); the introductions to these chapters will give a more detailed description of their contents. The other thread applies Lorentz geometry to special and general relativity (Chapters 6,12, and 13). The fact that relativity theory is expressed in terms of Lorentz geometry is lucky for geometers, who can thus penetrate surprisingly quickly into cosmology (redshift, expanding universe, and big bang) and, a topic no less interesting geometrically, the gravitation of a single star (perihelion precession, bending of light, and black holes). The tendency of the spacetimes in Chapters 12 and 13 to have singularities (big bang and black holes) is accounted for in abstract Lorentz terms by two theorems, due respectively to S. W. Hawking and R. Penrose; these are the goals of Chapter 14.

The general approach of the book is coordinate-free; however, coordinates are not neglected. Typically, geometric objects are defined invariantly and then described in terms of coordinates. In particular, the definition of a tensor I have adopted converts almost automatically into the classical coordinate formulation. A number of key proofs are given in classical notation. This attitude is only reasonable in view of the vast literature in each style.

The basic prerequisites for the book are modest: a good working knowledge of multivariable differential calculus, a firm belief in the existence and uniqueness theorems of ordinary differential equations, and an acquaintance with the fundamentals of point set topology and algebra. Later on, a knowledge of fundamental groups, covering spaces, and Lie groups is required; the necessary background in these topics is outlined briefly in Appendixes A and B. A college course in physics (particularly Newtonian mechanics) is required, not to read this book, but to appreciate the transformation and unification of Newtonian concepts effected by Einstein's relativistic geometry and the remarkable way the old and new theories—so different at base—reach approximate agreement on, say, the running of the solar system (Appendix C versus Chapter 13).

In the early chapters (1–5 and 7) the logical ordering is fairly strict. Thereafter the two branches— 8,9,11 and 6,12,13— are almost independent. (Chapters 12 and 13 require only an occasional reference to Chapters 9 and perhaps 8.) Chapter 10 is used in Chapters 11 and 14. Otherwise Chapter 14, though strongly motivated by Chapters 12 and 13, depends logically on only the early chapters.

Following each chapter are a number of exercises; these are meant to be workable without undue strain. In each chapter a single sequence of numbers designates collectively the theorems, lemmas, examples, and so on. For instance, Lemma 5.12 is the twelfth designated item in Chapter 5, not the twelfth lemma. Within a given chapter, the chapter number is omitted. Initials in square brackets, e.g., [SW], direct the reader to the References.

It is a pleasure to express my gratitude to the authors of the following brilliant and very different books: S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*; C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*; R. K. Sachs and H. Wu, *General Relativity for Mathematicians*.

NOTATION AND TERMINOLOGY

The following notations are among the most frequently used throughout the book:

M, N	manifolds	p, q	points
f, g, h	real-valued functions	α, β, γ	curves
v, w	vectors	V, W, X, Y	vector fields
ϕ, ψ	mappings	$\mathscr{U}, \mathscr{V}$	open sets

$\xi = (x^1, \ldots, x^n)$ coordinate system

$\boldsymbol{R}$ is the real number field, I denotes an open interval in $\boldsymbol{R}$, and, for example, $[a, b) = \{r \in \boldsymbol{R}: a \leq r < b\}$. The identity map is id; $\phi \circ \psi$ is the composite mapping that sends p to $\phi(\psi p)$. See Appendix B for Lie group notation such as $GL(n, \boldsymbol{R})$.

A mapping $\phi: M \to N$ is *one-to-one* (injective) if $p \neq q$ implies $\phi p \neq \phi q$. The *image* of ϕ is $\{\phi p : p \in M\} \subset N$, and ϕ is *onto* (surjective) if image $\phi = N$. (Inclusion $B \subset N$ does not exclude equality $B = N$.) If $B \subset N$ then $\phi^{-1}(B) = \{p \in M : \phi p \in B\}$, and when ϕ is one-to-one and onto, ϕ^{-1} also denotes the inverse mapping of ϕ.

If $\pi \circ \tilde{\phi} = \phi$, then $\tilde{\phi}$ is called a *lift* of ϕ through π. A lift of the identity map is called a *cross section* (or merely a *section*).

A *linear isomorphism* of vector spaces is a linear transformation that is one-to-one and onto, hence is *invertible.*

A subset A of a topological space has closure $\bar{A}$, interior int A, and boundary bd A.

CONTENTS

13. SCHWARZSCHILD GEOMETRY

14. CAUSALITY IN LORENTZ MANIFOLDS

APPENDIX B. LIE GROUPS

1 MANIFOLD THEORY

Generally speaking, a manifold is a topological space that locally resembles Euclidean space. A smooth manifold is a manifold M for which this resemblance is sharp enough to permit the establishment of partial differentiation—in fact, all the essential features of calculus—on M. Smooth manifolds are thus the natural setting for "calculus in the large."

SMOOTH MANIFOLDS

Euclidean n-space $\boldsymbol{R}^n$ is the set of all n-tuples $p = (p_1, \ldots, p_n)$ of real numbers. We assume in particular a familiarity with its structure as a vector space and as a topological space. The natural inner product of $\boldsymbol{R}^n$ is the *dot product* $p \cdot q = \sum p_i q_i$, with *norm* $|p| = \sqrt{p \cdot p}$. The resulting *metric* $d(p, q) = |p - q|$ is compatible with the topology of $\boldsymbol{R}^n$.

A real-valued function f defined on an open set $\mathscr{U}$ of $\boldsymbol{R}^n$ is *smooth* (or equivalently, C^∞) provided all mixed partial derivatives of f—of all orders—exist and are continuous at every point of $\mathscr{U}$.

For $1 \le i \le n$, let $u^i\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be the function that sends each point $p = (p_1, \ldots, p_n)$ to its ith coordinate p_i. Then $u^1, \ldots, u^n$ are the *natural coordinate functions* of $\boldsymbol{R}^n$.

A function ϕ from an open set $\mathscr{U}$ of $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$ is *smooth* provided each real-valued function $u^i \circ \phi$ is smooth ($1 \le i \le n$).

We can now make precise the resemblance to Euclidean space mentioned above. A *coordinate system* (or *chart*) in a topological space S is a homeomorphism ξ of an open set $\mathscr{U}$ of S onto an open set $\xi(\mathscr{U})$ of $\boldsymbol{R}^n$. If we write

$$\xi(p) = (x^1(p), \ldots, x^n(p)) \qquad \text{for each} \quad p \in \mathscr{U},$$

the resulting functions $x^1, \ldots, x^n$ are called the *coordinate functions* of ξ. Thus

$$\xi = (x^1, \ldots, x^n) \colon \mathscr{U} \to \boldsymbol{R}^n.$$

Here we call n the *dimension* of ξ. Note the identity $u^i \circ \xi = x^i$.

Two n-dimensional coordinate systems ξ and η in S *overlap smoothly* provided the functions $\xi \circ \eta^{-1}$ and $\eta \circ \xi^{-1}$ are both smooth. Explicitly, if $\xi \colon \mathscr{U} \to \boldsymbol{R}^n$ and $\eta \colon \mathscr{V} \to \boldsymbol{R}^n$, then $\eta \circ \xi^{-1}$ is defined on the open set $\xi(\mathscr{U} \cap \mathscr{V})$ and carries it to $\eta(\mathscr{U} \cap \mathscr{V})$—while its inverse function $\xi \circ \eta^{-1}$ runs in the opposite direction (see Figure 1). These functions are then required to be smooth in the usual Euclidean sense defined above. This condition is considered to hold trivially if $\mathscr{U}$ and $\mathscr{V}$ do not meet.

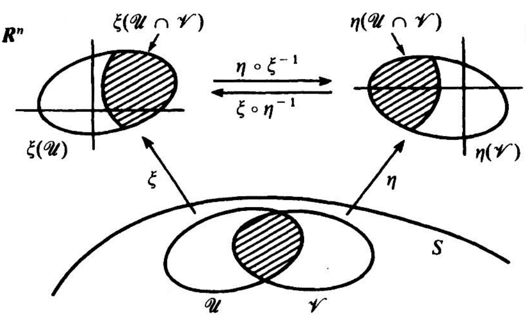

Figure 1.

1. Definition. An *atlas* $\mathscr{A}$ of dimension n on a space S is a collection of n-dimensional coordinate systems in S such that

(A1) each point of S is contained in the domain of some coordinate system in $\mathscr{V}$, and
(A2) any two coordinate systems in $\mathscr{A}$ overlap smoothly.

An atlas on S makes it possible to do calculus consistently on all of S. But different atlases may produce the same calculus, a technical difficulty eliminated as follows. Call an atlas $\mathscr{C}$ on S *complete* if $\mathscr{C}$ contains each coordinate system in S that overlaps smoothly with every coordinate system in $\mathscr{C}$.

2. Lemma. Each atlas $\mathscr{A}$ on S is contained in a unique complete atlas.

Proof. If $\mathscr{A}$ has dimension n, let $\mathscr{A}'$ be the set of all n-dimensional coordinate systems in S that overlap smoothly with every one contained in $\mathscr{A}$.

(a) *$\mathscr{A}'$ is an atlas* (of the same dimension as $\mathscr{A}$).

Since (A1) is obvious, consider (A2). If $\eta_1, \eta_2 \in \mathscr{A}'$, then by symmetry we need only prove that the function $\eta_1 \circ \eta_2^{-1}$ is Euclidean smooth. For any point $p \in \boldsymbol{R}^n$ in its domain, choose a $\xi \in \mathscr{A}$ whose domain contains $\eta_2^{-1}(p)$. As a composition of smooth functions, $(\eta_1 \circ \xi^{-1}) \circ (\xi \circ \eta_2^{-1})$ is smooth. Since this function equals $\eta_1 \circ \eta_2^{-1}$ on a neighborhood of p, the latter is smooth on that neighborhood. Smoothness being a local property, (a) follows.

(b) *$\mathscr{A}'$ is complete.* If a coordinate system ξ in S overlaps smoothly with every element of $\mathscr{A}' \supset \mathscr{A}$, then by definition $\xi \in \mathscr{A}'$.

(c) *$\mathscr{A}'$ is the unique complete atlas containing $\mathscr{A}$.*
If $\mathscr{C}$ is another, then since $\mathscr{C}$ contains $\mathscr{A}$, (A2) guarantees that $\mathscr{C} \subset \mathscr{A}'$. But then (A2) implies $\mathscr{A}' \subset \mathscr{C}$. ■

3. **Definition.** A *smooth manifold M* is a Hausdorff space furnished with a complete atlas.

There are many variants of the notion of manifold but for us *manifold* will mean *smooth manifold* as above. Any atlas $\mathscr{A}$ on a Hausdorff space makes it a manifold since we agree always to use the unique complete atlas containing $\mathscr{A}$ to fulfill Definition 3. The *dimension* $n = \dim M$ of a manifold M is the dimension of its atlas, and is often indicated by the notation M^n.

A coordinate system ξ *in a manifold M* is a coordinate system belonging to the complete atlas of M. If the domain $\mathscr{U}$ of ξ contains the point $p \in M$, then ξ is called a coordinate system *at p* and $\mathscr{U}$ a *coordinate neighborhood* of p.

If ξ is a coordinate system in M and $\mathscr{V}$ is an open set contained in the domain of ξ, then by completeness $\xi|\mathscr{V}$ is also a coordinate system in M.

4. **Examples of Manifolds.** (1) The identity map $(u^1, \ldots, u^n)$ of $\boldsymbol{R}^n$, by itself, is an atlas. From now on, $\boldsymbol{R}^n$ will denote the resulting n-dimensional manifold, called *Euclidean n-space.*

(2) *The sphere S^n.* Let S^n be the subspace $\{a \in \boldsymbol{R}^{n+1} : |a| = 1\}$ of $\boldsymbol{R}^{n+1}$. For each $1 \leq i \leq n + 1$, let $\mathscr{U}_i$ $[\mathscr{U}_i^-]$ be the open hemisphere consisting of all points a with $a_i > 0$ $[a_i < 0]$. The restriction to $\mathscr{U}_i$ or $\mathscr{U}_i^-$ of the coordinate functions $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^{n+1}$ gives a coordinate system in the space S^n. It is easy to check that the $2(n + 1)$ coordinate systems gotten in this way constitute an atlas on S^n making it an n-dimensional manifold.

(3) A two-dimensional manifold is often called a *surface*, and generally speaking, the objects called surfaces in elementary calculus (torus, cylinder, paraboloid, etc.) are two-dimensional manifolds.

We now consider two simple ways to get new manifolds from old.

Let $\mathscr{U}$ be an open set in a manifold M. Let $\mathscr{A}'$ be the set of all coordinates systems ξ in M such that the domain of ξ is contained in $\mathscr{U}$. By the remark preceding Example 4 these domains cover $\mathscr{U}$. Hence $\mathscr{A}'$ is an atlas on $\mathscr{U}$, making it a manifold called an *open submanifold* of M. Open sets of a manifold will always be considered to be open submanifolds.

If M and N are manifolds, let

$$\xi = (x^1, \ldots, x^m) : \mathscr{U} \to \boldsymbol{R}^m \qquad \text{and} \qquad \eta = (y^1, \ldots, y^n) : \mathscr{V} \to \boldsymbol{R}^n$$

be coordinate systems in M and N, respectively. The product function $\xi \times \eta : \mathscr{U} \times \mathscr{V} \to \boldsymbol{R}^{m+n}$ is defined by

$$(\xi \times \eta)(p, q) = (x^1(p), \ldots, x^m(p), y^1(q), \ldots, y^n(q)).$$

Evidently $\xi \times \eta$ is a coordinate system in the Hausdorff space $M \times N$, and it is easy to see that any two such *product coordinate systems* in $M \times N$ overlap smoothly.

5. Lemma. If M and N are manifolds, then the set of all product coordinate systems in $M \times N$ is an atlas on $M \times N$ making it the *product manifold* of M and N.

The dimension of $M \times N$ is $\dim M + \dim N$. This construction extends in an obvious way to the product of any finite number of manifolds. Indeed Euclidean space $\boldsymbol{R}^n$, as in Example 4, is exactly the product manifold $\boldsymbol{R}^1 \times \cdots \times \boldsymbol{R}^1$ (n factors).

SMOOTH MAPPINGS

Consider first the special case of a real-valued function f on a manifold M. If $\xi : \mathscr{U} \to \boldsymbol{R}^n$ is a coordinate system in M, then the composite function $f \circ \xi^{-1} : \xi(\mathscr{U}) \to \boldsymbol{R}^1$ is called the *coordinate expression* for f in terms of ξ. In fact,

$$f = (f \circ \xi^{-1})(x^1, \ldots, x^n) \quad \text{on} \quad \mathscr{U}.$$

(Compare, from elementary calculus, expressing a function $f = f(x, y)$ in terms of polar coordinates.) It is natural then to define a function $f : M \to \boldsymbol{R}$ to be *smooth* provided that for every coordinate system ξ in M the coordinate expression $f \circ \xi^{-1}$ is smooth in the usual Euclidean sense. Let $\mathfrak{F}(M)$ be the set of all smooth real-valued functions on M. If f and g are smooth functions on M so is their sum $f + g$ and product fg. The usual algebraic rules hold for these two operations, making $\mathfrak{F}(M)$ a commutative ring. Multiplicative inverses do not exist in general, but if $f \in \mathfrak{F}(M)$ is never zero, then $1/f \in \mathfrak{F}(M)$.

The notion of smoothness extends from a real-valued function to an arbitrary mapping of manifolds using the same idea: that coordinate expressions must be Euclidean smooth.