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Kolumban Hutter
Alphons A. F. van de Ven

Field Matter Interactions in Thermoelastic Solids

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Field Matter Interactions in Thermoelastic Solids

A Unification of Existing Theories
of Electro-Magneto-Mechanical Interactions

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To our wives
Barbara and Maria

PREFACE

The last two decades have witnessed a giant impetus in the formulation of electrodynamics of moving media, commencing with the development of the most simple static theory of dielectrics at large elastic deformations, proceeding further to more and more complex interaction models of polarizable and magnetizable bodies of such complexity as to include magnetic dissipation, spin-spin interaction and so on and, finally, reaching such magistral synthesis as to embrace a great variety of physical effects in a relativistically correct formulation. Unfortunately, the literature being so immense and the methods of approach being so diverse, the newcomer to the subject, who may initially be fascinated by the beauty, breadth and elegance of the formulation may soon be discouraged by his inability to identify two theories as the same, because they look entirely different in their formulation, but are suggested to be the same through the description of the physical situations they apply to.

With this tractate we aim to provide the reader with the basic concepts of such a comparison. Our intention is a limited one, as we do not treat the most general theory possible, but restrict ourselves to non-relativistic formulations and to theories, which may be termed deformable, polarizable and magnetizable thermoelastic solids. Our question throughout this monograph is basically: What are the existing theories of field-matter interactions; are these theories equivalent, and if so; what are the conditions for this equivalence?

We are not the first ones to be concerned with such fundamental ideas. Indeed, it was W.F. Brown, who raised the question of non-uniqueness of the formulation of quasistatic theories of magnetoelastic interactions, and within the complexity of his theory, he could also resolve it. Penfield and Haus, on the other hand, were fundamentally concerned with the question how electromagnetic body force had to be properly selected. This led them to collect their findings and to compare the various theories in an excellent monograph, in which they rightly state that equivalence of different formulations of electrodynamics of deformable continua cannot be established without resort to the constitutive theory, but at last, they dismissed the proper answer, as their treatment is incomplete in this regard. For this reason the entire matter was re-investigated in the doctoral dissertation of one of us (K. Hutter), but this work was soon found unsatisfactory and incomplete in certain points, although the basic structure of the equivalence proof as given in Chapter 3 of this tractate, was essentially already outlined there. Moreover, Hutter was still not able to compare certain magnetoelastic interaction theories so that what he attempted remained a torso anyhow.

The difficulties were overcome by van de Ven in a series of letters, commencing in fall of 1975, in which we discussed various subtleties of magnetoelastic interactions that had evolved from each of our own work. The correspondence was so fruitful that we soon decided to summarize our efforts in a joint publication. It became this monograph, although this was not our initial intention. Yet, after we realized that a proper treatment required a presentation at considerable length, we decided to be a little broader than is possible in a research report and to write a monograph, which would be suitable at least as a basis for an advanced course in continuum mechanics and electrodynamics (graduate level in the US). We believe that with this text this goal has been achieved. We must at the same time, however, warn the reader not to take this tractate as a basis to learn continuum mechanics and/or electrodynamics from the start. The fundamentals of these subjects are assumed to be known.

Our acknowledgements must start with mentioning Profs.J.B. Alblas (Technological University Eindhoven) and Y.H. Pao (Cornell University). They were the ones who initiated our interest in the subject of magnetoelastic interactions. While performing the research for this booklet and during our preparation of the various draughts we were supported by our institutions, the Federal Institute of Technology, Zürich and the Technological University, Eindhoven, and were, furthermore, encouraged by Prof. J.B. Alblas, Eindhoven, Prof. D. Vischer, Zürich, Prof. I. Müller, Paderborn, Prof. H. Parkus, Vienna, Dr. Ph. Boulanger, Brussels and Dr. A. Prechtl, Vienna. The support and criticism provided by them, directly or indirectly, were extremely helpful. We are grateful to these people not only for their keen insight and willingness to discuss the issues with us, but also for their encouragement in general.

During the initial stage and again towards the end of the write-up of the final draught of this monograph K. Hutter was financially supported in parts by the Technological University, Eindhoven, to spend a total of a two months period (September 1976 and April 1978) at its Mathematics Department. Without the hospitality and the keen friendship of the faculty and staff members of this department and especially of Prof. J.B. Alblas and his group, the work compiled in these notes would have barely have been finished so timely. The burden of typing the manuscript was taken by Mrs. Wolfs-Van den Hurk. It was her duety to transform our hand-written draughts into miraculously looking typed sheets of over 200 pages. Her effort, of course, is gratefully acknowledged.

Eindhoven and Zürich
in the summer of 1978

K. Hutter
A.A.F. van de Ven

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1. BASIC CONCEPTS

1.1 PREVIEW

With the boundaries of physical science expanding rapidly, present day engineers must necessarily assimilate information and knowledge of subjects that continue to become more and more complex. In the study of the nature and mechanical behavior of engineering materials, however, their knowledge seldom progresses beyond the level of elementary theory of elasticity and simple ideas of the theory of plasticity. More esoteric theories of material behavior and the interaction with various fields are generally left out of consideration or soon abolished as being mathematically intractable or economically unjustifyable.

This book is an account on one of the above mentioned more esoteric theories. What we have in mind is the response of deformable bodies to electromagnetic fields. Indeed, the interaction of electromagnetism with thermoelastic fields is not only a challenging scientific problem, but it is increasingly attracting also engineers in the nuclear power and electronic industry from a purely applied point of view.

The subject of electrodynamics of moving media has always been a controversal one. This book will not end or resolve all controversies, because we can answer some, but not all the relevant questions in connection with a complete thermodynamic theory of electromagnetism.

The basic difficulties in the description of electromechanical interaction models are manifold. A first difficulty is concerned with the invariance properties of the electromagnetic field equations. As is well-known, Maxwell's equations are invariant under Lorentz transformations, while the balance laws of classical mechanics are invariant under Euclidean transformations and frame indifferent under Galilei transformations. Clearly, a proper derivation should also treat the mechanical equations relativistically. This is true, but for most problems of technical relevance, relativistic effects are negligible.

It is therefore customary, in general, to treat the mechanical equations classically, while the equations of electrodynamics are handled relativistically. In so doing it might in these theories become uncertain what transformation properties some variables are based upon. Yet the knowledge of such transformation properties is important, because they give us indications as to what variables are comparable among different theories.

A second and even more serious difficulty can be found in the definitions of electromagnetic body force, body couple and energy supply. The roots of this difficulty lie in the separation of the electromagnetic field quantities in near and far field effects. This separation has been and still is the root of

controversies, because almost every author separates the total fields differently. In other words, near and far fields are not unique.
A third difficulty is connected with the Maxwell equations, which for deformable moving matter were first derived by Minkowski. Apart from the Maxwell equations in Minkowski's form there exists a variety of other forms of the Maxwell equations in deformable media, all of which are motivated from particular models. The "action" of the electromagnetic fields upon the material is described hereby by quantities referred to as polarization and magnetization. However, dependent upon the model of derivation, polarization and magnetization of one theory may be and in general are different from polarization and magnetization of another theory. Hence, while all formulations of electromagnetism of deformable continua are equally valid - we know of five different descriptions - special care must be observed that variables of one theory are not confused with those of another.
As a final difficulty, we mention that the equations of a theory of electromechanical interaction are highly nonlinear. Generally, they defy any exact analysis even for the most simple problems that are of physical relevance. As a result, linearization procedures are needed.
To render the above statements more precise, consider the equations of motion which may be derived by formulating the balance law of momentum to an arbitrary part of the body. The local form of this balance law states that "mass times acceleration equals divergence of stress plus body force". In ordinary classical mechanics the body force is either set equal to zero, or else given by the gravitational force. A body couple hardly occurs in applications, in which case the balance law of moment of momentum implies the symmetry of the (Cauchy) stress tensor. When the body under consideration is interacting with electromagnetic fields, however, body force and body couple are given by electromagnetic quantities.
The total force and the total moment excerted on a body by electromagnetic fields may be separated into a long range and a short range effect. The long range effect is expressed as a body force and body couple. The short range effect, on the other hand, manifests itself as surface tractions which can be combined with the mechanical tractions giving rise thereby to the definition of the stress tensor. This decomposition is not unique, thus leading to non-unique body force expressions and non-unique stress tensors. As a consequence, the electromagnetic body couple cannot be unique either.
Although this non-uniqueness might be quite striking to the novel reader it is nontheless not disturbing at all if looked upon from the right point of view.

Indeed, it is not important that the above separation into force and stress is unique, because differences in the body forces can always be absorbed in the stress tensors, provided that they are expressible as a divergence of a stress. A variety of mutually incompatible formulas for the force expressible in terms of stresses are therefore equivalent with respect to the total force. Only this force is physically observable. Thus the incompatibility is not physical, but metaphysical or semantic.

The incompatibility expressed above also occurs in the energy equation (first law of thermodynamics). This equation states that the time rate of change of the internal energy is balanced by the power of working of the stresses, the divergence of the heat flux and the energy supply due to electromagnetic effects and due to heat. Since stress was already said to be non-unique, it follows that internal energy, heat flux and electromagnetic energy supply cannot be determined uniquely either. Likewise, the electromagnetic energy supply might contain a term that is the divergence of a vector which could be absorbed in the heat flux vector. As an immediate consequence, it cannot be assured that heat flux is energy flux of thermal nature. We shall therefore prefer the term energy flux instead.

As was the case for the momentum equation, seemingly incompatible expressions for internal energy, energy flux and energy supply of electromagnetic origin do not prevent two theories from being equivalent. However, it is easily understandable that a proof of equivalence must be difficult in general for, stress, internal energy and entropy (and also some electromagnetic field vectors) are interrelated by thermodynamic conditions. More explicitly, thermodynamic requirements make stress and entropy (and other quantities) derivable from a so called free energy. If two theories are different in the body force, body couple and energy supply, therefore, the condition that the momentum equation and energy equation remains the same must amount to an interrelation between the free energies of two theories. Hence, equivalence of two theories of electromechanical interactions is a thermodynamic statement in general.

The question of equivalence of two theories lies at the center of the different formulations of electromechanical interaction theories. Although there is a valid point behind the statement that equivalence of different theories need not be proved, because these theories describe different physical situations, we nevertheless take the position that different formulations of electromechanical interaction theories should yield the same results for physically measurable quantities, if the theories are claimed to be applicable to a certain class of material response. For instance, if we call a material a thermoelastic polarizable and magnetizable solid and if there is more than one formulation for such a solid, one should expect that, irrespective of all differences in the details, these theories will in any initial boundary value problem deliver the same results for physically measurable quantities. Measurable or observable quantities are all those which can be measured uniquely by two different observers. All kinematical quantities that are derivable from the motion are measurable in principle and so is the (empirical) temperature. Regarding electromagnetic field quantities, we take the position that they are not measurable except in vacuo where they can be observed by measuring the force on a test charge. There exist variables not observable by any means. These are all those which are not defined except by the mathematical properties laid down for them.

To demonstrate the equivalence of the different formulations of electromechanical interaction theories it is necessary to prove that physically measurable quantities in two formulations assume the same values in every point of the body for any initial boundary value problem. This does not only mean that the field equations of one theory must be transformable into those of the other, but this condition also includes the boundary and initial conditions. One of the major goals of this monograph is to give an exposition of the existing theories of polarizable and magnetizable electrically and thermally conducting materials and to show in what sense they can be called equivalent.

The reasons behind this non-uniqueness of electrodynamics in moving media are twofold. For one, the action of the body on the electromagnetic fields is generally described by adding to the field variables occurring in vacuo two other electromagnetic field vectors. This addition is not unique and results in different forms of the Maxwell equations. Second, even when we restrict ourselves to a particular form of the Maxwell equations, the electromagnetic forces, couples and energy supply terms need not be unique. More precisely, we mention that the two electromagnetic field vectors describing the interaction of a ponderable body can be introduced for instance by postulating that every material point is equipped with a number of non-interacting electric and magnetic dipoles. These dipole moments then form the two additional electromagnetic field vectors which are called polarization and magnetization. When the calculations with these postulated dipoles are carried through consistently, a certain set of Maxwell equations (now called the Chu-formulation) emerges. These equations are different from those which follow from the postulation that magnetization is modeled as an electric circuit which follows the motion of the material particle in question (statistical and Lorentz formulations).

As far as electromagnetic body forces are concerned these are not even unique when one is restricting oneself to a particular interaction model. Indeed, in the Chu-formulation we shall present two versions of body force expressions and we shall prove that both are not distinguishable by any measurements. This proof will also be given for all other formulations. However, we shall not present the models as such, because they are amply treated in the pertinent literature.

Although the proof of the equivalence of various theories of electrodynamics in deformable continua is a very important achievement, we want to state here clearly that we have performed this proof only on the level of non-relativistic theories. The exact definitions of the term "non-relativistic" will be made precise in the respective Chapters. It may suffice to mention that it essentially means that in MKSA-units terms containing a c^{-2}-factor are neglected. Here, c is the speed of light in vacuo. There exists a number of other theories of electromechanical interactions in which it is claimed that terms of order V^2/c^2 are neglected (V = velocity of the particle in the body) while those containing a c^{-2}-factor are kept. We term such approximations "semi-relativistic". Quasistatic theories (terms, containing a c^{-1}-factor are neglected) will not be treated here.

It is a well-known fact that fluids are best handled in the spatial description. It is also known that electrodynamics is usually only formulated in the spatial description. Yet for a theory of solids it would be advantageous when all equations could be referred to the reference configuration. This is indeed possible and it essentially amounts to the introduction of new electromagnetic field variables. It turns out that these so called Lagrangian field variables are much more convenient to describe the theory of solids, because many thermodynamic formulas appear in a more condensed form this way. Another reason for the introduction of the material description is its advantage in the linearization of the governing equations. This linearization procedure is substantially easier when performed in the material rather than in the spatial description.

This brings us naturally to the linearization procedure of the various theories. In principle, there are two alternatives open to extract some useful information from these complicated equations. One is to find numerical solutions for the nonlinear equations and the other is to linearize the equations on the basis of a sequence of consistent approximations. We shall follow the latter, because it provides a better access to the real physics of the problem. The linearization procedure is analogous to situations referred to as "small fields superimposed upon large fields". The difference between these general

treatments and ours is that the deformations are assumed to be small. This assumption is not necessary, and indeed the formal expansion procedures we shall apply also hold true for the general case. When the restriction to small deformations is used, however, it means physically that large external fields primarily induce strong electromagnetic fields within the body, but only small deformations. Therefore in the first step of evaluating the induced electromagnetism, the deformations may be neglected alltogether.
A set of zeroth order equations which formally agrees with rigid body electrodynamics, is thus obtained. In the second step small strains are considered which add small but important corrections to the zeroth order electromagnetic fields. Thus, the second set consists of linear field equations, the coefficients of which generally depend upon the zeroth order electromagnetic fields. These field equations may then be applied to solve problems like magnetoelastic buckling, wave propagation in a material subject to electromagnetic fields, etc.
Clearly, because we shall prove that all electromechanical interaction theories of polarizable and magnetizable solids are equivalent, the linearization procedure mentioned above need only be performed for one particular theory which can be selected according to our needs. Moreover, it should be clear that this equivalence must amount in the statement how the free energy as a function of its independent variables in one theory is related to the free energy of another theory. The set of independent variables in this second free energy may very well be different from the first one. In other words, the correspondence relations for equivalence of various theories are dependent on which set of independent variables is chosen in the constitutive relations, but it is quite clear that the equivalence as such should not depend on the choice of the independent fields. From a theoretical point of view the problem just raised is not a serious one, because, in principle, equivalence of different constitutive formulations in one single theory can be established quite easily. It then suffices to prove equivalence of two different electromagnetic descriptions of deformable bodies with the aid of just one constitutive formulation in each of them.
The problem to find the free energy of a particular formulation from that of another one is a very difficult problem in practice, however. It amounts to solving a functional differential equation the solutions to which are not known to date. Nevertheless, special cases are straightforward to handle. They serve as explicit examples which should demonstrate that equivalence is possible. Mathematically this is important, because it serves as an explicit

demonstration that the functional differential equations mentioned above do admit exact solutions. That these correspond to a reasonable physical situation is a nice additional property. The general question of existence and nonexistence of solutions will not be attacked here. Instead we look at the approximations in the way described below.

It is customary in applications to write for the free energies polynomial expressions, and it is generally assumed that these polynomials can be truncated at a certain level. When polarization and magnetization are amongst the independent constitutive variables the free energy will be a polynomial expression in the deformation tensor, the temperature, polarization and magnetization, and the coefficients in this polynomial expression give rise to effects like magnetic and electric anisotropy, magnetostriction, electrostriction etc. The coefficients bear the names electric and magnetic susceptibilities etc. The same theory could be derived also with the electric field strength and the magnetic induction as independent fields instead of magnetization and polarization. The free energy of this formulation would again be expressed as a polynomial of its variables and it would again be truncated at a certain level. This polynomial would again give rise to effects like electric and magnetic anisotropies, magnetostriction and electrostriction etc, but it is evident that the coefficients of this polynomial must be different from those of the other, if the two formulations aim at describing the same phenomena. The literature is full of confusion in this regard, mainly because different coefficients bear the same name. From the above it is, however, quite clear that there must be relations between the above mentioned coefficients. We shall show what these relations look like and in what sense the emerging approximate theories can be regarded to be equivalent. The findings can be summarized as follows: Two formulations, in which the free energies are represented by polynomials of a certain order in their variables can only be equivalent to within terms that were omitted in the expansion process. Only on the basis that these terms are negligibly small can we claim two theories to be equivalent. An analogous statement also holds for one single formulation in which certain constitutive quantities are interchanged as dependent and independent variables.

We end with an outlook on problems this theory may be applied to.

1.2 KINEMATICS

As is common in continuum mechanics, we regard a body as a three-dimensional manifold embedded in Euclidean 3-space. Its elements are called particles. Let $\mathcal{R}_R$ be its reference configuration and $\mathcal{R}_t$ its configuration at time t. Instead of $\mathcal{R}_t$ we shall subsequently write $\mathcal{R}$, and we shall refer to $\mathcal{R}$ as the present configuration of the body. Parts of the body will be denoted by V_R and $\mathcal{V}$, dependent on whether they are referred to the reference configuration and the present configuration, respectively. The boundary of V_R and of $\mathcal{V}$ will be denoted by ∂V_R and $\partial\mathcal{V}$, respectively. The position of a particle in $\mathcal{R}_R$ will be designated by $\underline{X}$ (X_α, $\alpha = 1,2,3$), whereas the one in the present configuration $\mathcal{R}$ is $\underline{x}$ (x_i, $i = 1,2,3$). A motion of the body is then described by the mapping

$$(1.1) \qquad x_i = \chi_i(X_\alpha, t), \qquad (i,\alpha = 1,2,3) .$$

We assume $\underline{\chi}$ to be invertible and this is tantamount to assuming that the functional determinant

$$(1.2) \qquad J := \det(\partial\chi_i/\partial X_\alpha) ,$$

never vanishes and may without loss of generality be assumed to be positive. In the above and throughout this monograph symbolic and Cartesian tensor notation is used. Greek indices refer to the material coordinates $\underline{X}$, and Latin indices to the spatial coordinates $\underline{x}$. Summation convention will be used over doubly repeated indices and commas preceding indices indicate differentiations with respect to space variables. The symbol d/dt or the superimposed dot will designate differentiation with respect to time t, holding the particle $\underline{X}$ fixed, i.e.

$$(1.3) \qquad \frac{d\Phi}{dt} \equiv \dot{\Phi} := \frac{\partial\Phi(\underline{X},t)}{\partial t} .$$

$\dot{\Phi}$ is called the material time derivative. Likewise, $\partial/\partial t$ will denote differentiation with respect to time t, holding the spatial position $\underline{x}$ fixed. Hence,

$$(1.4) \qquad \frac{\partial\Phi}{\partial t} := \frac{\partial\Phi(\underline{x},t)}{\partial t} .$$

$\partial\Phi/\partial t$ is called the local, or partial, time derivative, and it is easy to see that (1.3) and (1.4) may be combined to yield

$$(1.5) \qquad \frac{d\Phi}{dt} = \dot{\Phi} = \frac{\partial\Phi}{\partial t} + \dot{x}_i \frac{\partial\Phi}{\partial x_i} .$$