

Sergei O. Kuznetsov
Stefan Schmidt (Eds.)

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Formal Concept Analysis

5th International Conference, ICFCA 2007
Clermont-Ferrand, France, February 2007
Proceedings



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Preface

This volume contains the Proceedings of ICFCA 2007, the Fifth International Conference on Formal Concept Analysis. The ICFCA conference series intends to serve as a distinguished forum for state-of-the-art research from foundational to applied lattice theory and related fields, all of which involve methods and techniques in the realm of FCA. We believe this year's conference provided a continuation of the high standard of its predecessors.

Formal concept analysis ranges from restructuring general algebra and lattice theory, on the one hand, to providing high-level conceptual analysis and knowledge processing techniques in various applied disciplines, on the other hand. This is due to the fact that FCA allows transforming relational data into implications and dependencies and, vice versa, exploring plausible hypotheses, suggested implications, or expected dependencies against data. This all is connected with the investigation of hierarchical aspects of information—based on a mathematical formalization of *conceptual hierarchy*. The field of FCA has been developing way beyond its original scope and attracts a still-growing number of sophisticated research scholars, who vary from theoreticians to more practical problem solvers. This volume reflects the diversity between fundamental methods and applied techniques, explored by an enthusiastic research community. Here, we only mention research areas such as data visualization, information retrieval, machine learning, data analysis and knowledge management.

ICFCA 2007 covered practical data analysis and problem solving as well as foundational progress of the field. The algorithmic side was in balance with theoretical discoveries. All regular papers in this volume were refereed by independent domain experts, although the Program Chairs are responsible for the final decision on publications in this volume. To assure a high-quality standard, the close involvement of the Program Committee and the Editorial Board in any decision making was crucial.

The General Conference Chair of ICFCA 2007, held in Clermont-Ferrand, France, was Lhouari Nourine. The success of the conference was a result of all involved, in particular, the General Conference Chair together with the Conference Organizing Committee, and the Editorial Board in adjunction with the Program Committee—to all of whom we express our warmest thanks. The activity of the first Program Co-chair in preparing the volume was supported by the DFG project “Concepts and Models” (COMO).

February 2007

Sergei O. Kuznetsov
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Table of Contents

Relational Galois Connections	1
<i>Bernhard Ganter</i>	
Semantology as Basis for Conceptual Knowledge Processing	18
<i>Peter Eklund and Rudolf Wille</i>	
A New and Useful Syntactic Restriction on Rule Semantics for Tabular Datasets	39
<i>Marie Agier and Jean-Marc Petit</i>	
A Proposal for Combining Formal Concept Analysis and Description Logics for Mining Relational Data	51
<i>Mohamed Hacene Rouane, Marianne Huchard, Amedeo Napoli, and Petko Valtchev</i>	
Computing Intensions of Digital Library Collections	66
<i>Carlo Meghini and Nicolas Spyrtos</i>	
Custom Asymmetric Page Split Generalized Index Search Trees and Formal Concept Analysis	82
<i>Ben Martin and Peter Eklund</i>	
The Efficient Computation of Complete and Concise Substring Scales with Suffix Trees	98
<i>Sébastien Ferré</i>	
A Parameterized Algorithm for Exploring Concept Lattices	114
<i>Peggy Cellier, Sébastien Ferré, Olivier Ridoux, and Mireille Ducassé</i>	
About the Lossless Reduction of the Minimal Generator Family of a Context	130
<i>Tarek Hamrouni, Petko Valtchev, Sadok Ben Yahia, and Engelbert Mephu Nguifo</i>	
Some Notes on Pseudo-closed Sets	151
<i>Sebastian Rudolph</i>	
Performances of Galois Sub-hierarchy-building Algorithms	166
<i>Gabriela Arévalo, Anne Berry, Marianne Huchard, Guillaume Perrot, and Alain Sigayret</i>	
Galois Connections Between Semimodules and Applications in Data Mining	181
<i>Francisco J. Valverde-Albacete and Carmen Peláez-Moreno</i>	

On Multi-adjoint Concept Lattices: Definition and Representation
Theorem 197
Jesús Medina, Manuel Ojeda-Aciego, and Jorge Ruiz-Calviño

Base Points, Non-unit Implications, and Convex Geometries 210
Bernhard Ganter and Heiko Reppe

Lattices of Relatively Axiomatizable Classes 221
Dmitry E. Pal'chunov

A Solution of the Word Problem for Free Double Boolean Algebras 240
Björn Vormbrock

On the MacNeille Completion of Weakly Dicomplemented Lattices 271
Léonard Kwuida, Branimir Seselja, and Andreja Tepavčević

Polynomial Embeddings and Representations 281
Tim Becker

The Basic Theorem on Labelled Line Diagrams of Finite Concept
Lattices 303
Rudolf Wille

Bipartite Ferrers-Graphs and Planar Concept Lattices 313
Christian Zschalig

Author Index 329

Relational Galois Connections

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Abstract. Galois connections can be defined for lattices and for ordered sets. We discuss a rather wide generalisation, which was introduced by Weiqun Xia and has been reinvented under different names: Relational Galois connections between relations. It turns out that the generalised notion is of importance for the original one and can be utilised, e.g., for computing Galois connections.

The present paper may be understood as an attempt to bring together ideas of Wille [15], Xia [16], Domenach and Leclerc [3], and others and to suggest a unifying language.

1 Galois Connections Between Relations

It is usual to define a **Galois connection** between two complete lattices (L_1, \leq_1) and (L_2, \leq_2) as a pair (φ, ψ) of mappings

$$\varphi : L_1 \rightarrow L_2, \quad \psi : L_2 \rightarrow L_1$$

satisfying for all $x \in L_1$ and all $y \in L_2$

$$x \leq_1 \psi(y) \iff y \leq_2 \varphi(x).$$

It is well known that if this condition is satisfied, both mappings φ and ψ are order reversing and that their compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are closure operators on L_2 and L_1 , respectively, with dually isomorphic lattices of closed sets (see, e.g., [4]).

Not so obvious is how to *construct* Galois connections for given lattices (L_1, \leq_1) and (L_2, \leq_2) . We shall address this question later. For the case that both lattices are power set lattices, the answer was given by Birkhoff [2]: Let G and M be sets, let $I \subseteq G \times M$ be some relation. Define

$$A^I := \{m \in M \mid g I m \text{ for all } g \in A\} \quad \text{if } A \subseteq G$$

and

$$B^I := \{g \in G \mid g I m \text{ for all } m \in B\} \quad \text{if } B \subseteq M.$$

Then

$$\varphi(X) := X^I \quad \text{for } X \subseteq G \quad \text{and} \quad \psi(Y) := Y^I \quad \text{for } Y \subseteq M$$

defines a Galois connection between the power set lattice of G and the power set lattice of M . The set

$$\mathfrak{B}(G, M, I) := \{(A, B) \mid A \subseteq G, B \subseteq M, A^I = B, A = B^I\},$$

ordered by

$$(A_1, B_1) \leq (A_2, B_2) : \Longleftrightarrow A_1 \subseteq A_2 \Longleftrightarrow B_2 \subseteq B_1$$

is a complete lattice, called the **concept lattice**¹. The sets of the form A^I , $A \subseteq G$, are called the **intents** of (G, M, I) , and those of the form B^I , $B \subseteq M$ are the **extents**. These are the closed sets of the two closure operators.

The notion of a Galois connection can be generalised to ordered sets and, even further, to arbitrary binary relations $I \subseteq G \times M$, $J \subseteq H \times N$, as in the next definition:

Definition 1. A **Galois connection**² between (G, M, I) and (H, N, J) is a pair (φ, ψ) of mappings

$$\varphi : G \rightarrow N, \quad \psi : H \rightarrow M$$

satisfying

$$g \, I \, \psi(h) \Longleftrightarrow h \, J \, \varphi(g).$$

This definition is symmetric: if (φ, ψ) is a Galois connection between (G, M, I) and (H, N, J) , then (ψ, φ) is a Galois connection between (H, N, J) and (G, M, I) . This corresponds to the original, *contravariant* definition of Galois connections. Some authors consider also the *covariant* version, which allows for composition of Galois connections. This is achieved when (H, N, J) is replaced by the *dual context* (N, H, J^{-1}) . These mappings are closely related to *infomorphisms* and to *Chu morphisms*, see Section 6 for more.

One might argue that this definition deviates from the original one for lattices or ordered sets. But it is only a natural generalisation. For two ordered sets (P, \leq_1) and (Q, \leq_2) the condition that (φ, ψ) is a Galois connection between (P, P, \leq_1) and (Q, Q, \leq_2) is

$$x \leq_1 \psi(y) \Longleftrightarrow y \leq_2 \varphi(x),$$

as usual.

We may generalise even further, replacing the pair of mappings by a pair of relations $\Phi \subseteq G \times N$ and $\Psi \subseteq H \times M$. The natural condition then is that

$$g \, I \, h^\Psi \Longleftrightarrow h \, J \, g^\Phi$$

holds for all $g \in G$ and all $h \in H$. We call this the (relational) **Galois condition**. However, this condition by itself turns out to be not strong enough. We therefore define

¹ Our notation is that of Formal Concept Analysis [5], where (G, M, I) is called a **formal context**. Other authors use names like **classification** [1], **Chu-space** [11], etc.

² Called **context–Galois connection** by Xia [16].

Definition 2. A **relational Galois connection**³ between (G, M, I) and (H, N, J) is a pair (Φ, Ψ) of relations

$$\Phi \subseteq G \times N \quad \text{and} \quad \Psi \subseteq H \times M$$

satisfying

1. $g I h^\Psi \iff h J g^\Phi$ for all $g \in G, h \in H$ (the Galois condition),
2. Φ is the largest relation satisfying the Galois condition for the given Ψ , and conversely.

	M	N
G	I	Φ
H	Ψ	J

Fig. 1. The Galois condition requires that the left hand part h^Ψ of a row from the lower part is contained in a row g^I of the upper part iff the right hand side g^Φ of the latter is contained in the former, in h^J

Proposition 1. The second condition of Definition 2 can be reformulated as follows:

- 2'. g^Φ is an intent of (H, N, J) and h^Ψ is an intent of (G, M, I) .

Proof. Fix $\Psi \subseteq H \times M$. A relation Φ satisfies the direction “ \Rightarrow ” of the Galois condition iff

$$g^\Phi \subseteq \{h \in H \mid g I h^\Psi\}^J.$$

The implication “ \Leftarrow ” is equivalent to

$$g^{\Phi J} \subseteq \{h \in H \mid g I h^\Psi\},$$

which implies

$$g^\Phi \subseteq \{h \in H \mid g I h^\Psi\}^J = g^{\Phi J J}.$$

Thus from Φ we obtain another relation satisfying the Galois condition for Ψ by replacing each g^Φ by its closure $g^{\Phi J J}$. This then is the largest possible choice. The dual argument works for Ψ . \square

The next proposition shows that this is in fact a further generalisation.

³ Called **essential Galois bond** by Xia [16]. Xia’s condition (iii) is implied.

Proposition 2. *If (φ, ψ) is a Galois connection between (G, M, I) and (H, N, J) then (Φ, Ψ) , given by*

$$g^\Phi := \varphi(g)^{JJ} \text{ for } g \in G, \quad h^\Psi := \psi(h)^{II} \text{ for } h \in H,$$

is a relational Galois connection.

The proof of Proposition 1 shows that we get the second condition essentially “for free”:

Proposition 3. *Whenever $R \subseteq G \times N$ and $S \subseteq H \times M$ are relations with*

$$g \ I \ h^S \iff h \ J \ g^R$$

then there is a unique relational Galois connection (Φ, Ψ) such that

$$g \ I \ h^S \iff g \ I \ h^\Psi \iff h \ J \ g^\Phi \iff h \ J \ g^R.$$

Proof. Define Φ and Ψ as follows: for $g \in G$ let $g^\Phi := (g^R)^{JJ}$, and for $h \in H$ let $h^\Psi := (h^S)^{II}$. \square

The next observation shows that our “radical” generalisation of Galois connections leads back to the original definition:

Lemma 1. *If (ϕ, ψ) is a Galois connection between concepts lattices $\underline{\mathfrak{B}}(G, M, I)$ and $\underline{\mathfrak{B}}(H, N, J)$, then*

$$g^\Phi = Y : \iff \phi(g^{II}, g^I) = (Y^J, Y)$$

and

$$h^\Psi = X : \iff \psi(h^{JJ}, h^J) = (X^I, X)$$

defines a relational Galois connection between (G, M, I) and (H, N, J) .

Conversely we obtain from each relational Galois connection (Φ, Ψ) between (G, M, I) and (H, N, J) a Galois connection (ϕ, ψ) between the concept lattices $\underline{\mathfrak{B}}(G, M, I)$ and $\underline{\mathfrak{B}}(H, N, J)$ by

$$\phi(X, X^I) := (X^{\Phi J}, X^\Phi) \quad \text{and} \quad \psi(Y, Y^J) := (Y^{\Psi I}, Y^\Psi).$$

The two constructions are inverse to each other.

Proof. The proof is straightforward. \square

The following proposition is no surprise:

Proposition 4. *In a relational Galois connection, the two parts determine each other. More precisely, if (Φ_1, Ψ) and (Φ_2, Ψ) are relational Galois connections between (G, M, I) and (H, N, J) , then $\Phi_1 = \Phi_2$, and dually.*

Proof. For each $g \in G$, we have

$$(g^{\Phi_1})^J = \{h \in H \mid g \ I \ h^\Psi\} = (g^{\Phi_2})^J$$

and therefore $g^{\Phi_1} = g^{\Phi_2}$. \square

2 G-Relations and Dual Bonds

We have seen in Proposition 4 that it suffices to study only one of the two parts of a relational Galois connection. So let us call $\Phi \subseteq G \times N$ a **G-relation** from (G, M, I) to (H, N, J) if there is some Ψ such that (Φ, Ψ) is a relational Galois connection (this term was coined by Xia [16] following Shmueli's notion of a G-ideal [12,13]). A G-relation is, loosely spoken, one half of a relational Galois connection.

Proposition 5. *A relation $\Phi \subseteq G \times N$ is a G-relation from (G, M, I) to (H, N, J) if and only if it satisfies*

1. for each $g \in G$, g^Φ is an intent of (H, N, J) ,
2. for each $h \in H$, $\{g \in G \mid h J g^\Phi\}$ is an extent of (G, M, I) .

Proof. If (Φ, Ψ) is a relational Galois connection then the two conditions obviously have to be satisfied. Conversely, assume 1.) and 2.) and define Ψ by

$$h^\Psi := \{g \in G \mid h J g^\Phi\}^I.$$

Clearly this is an intent and we have

$$\begin{aligned} g I h^\Psi &\iff g \in (h^\Psi)^I \\ &\iff g \in \{x \in G \mid h J x^\Phi\}^{II} \\ &\iff g \in \{x \in G \mid h J x^\Phi\} \\ &\iff h J g^\Phi. \end{aligned} \quad \square$$

Note that the set of relation pairs satisfying the Galois condition is closed under (arbitrary) unions:

$$g I h^{\Psi_1} \iff h J g^{\Phi_1} \quad \text{and} \quad g I h^{\Psi_2} \iff h J g^{\Phi_2}$$

together imply

$$g I h^{\Psi_1 \cup \Psi_2} \iff h J g^{\Phi_1 \cup \Phi_2}.$$

Therefore, the relations $\Phi \subseteq G \times N$ for which there is some $\Psi \subseteq H \times M$ satisfying the Galois condition, ordered by inclusion, form a complete lattice. Moreover, Proposition 3 guarantees that for each such relation there is a smallest G-relation containing it (its **G-closure**⁴). As a consequence, we obtain

Theorem 1 (Xia [16]). *The G-relations from (G, M, I) to (H, N, J) , ordered by set inclusion, form a complete lattice. The supremum of G-relations is the G-closure of their union.*

This lattice is isomorphic to the complete lattice of Galois connections between the corresponding concept lattices $\mathfrak{B}(G, M, I)$ and $\mathfrak{B}(H, N, J)$.

An example is given in Figure 4.

Another encoding of a relational Galois connection in only one relation is analogous to Birkhoff's construction:

⁴ Note, however, that G-relations are not necessarily closed under intersections.

Definition 3. A dual bond⁵ from (G, M, I) to (H, N, J) is a relation $R \subseteq G \times H$ for which it holds that

- for every $g \in G$, g^R is an extent of (H, N, J) and
- for every $h \in H$, h^R is an extent of (G, M, I) .

Theorem 2. From each relational Galois connection (Φ, Ψ) between (G, M, I) and (H, N, J) we obtain a dual bond $R \subseteq G \times H$ by means of

$$g R h \quad : \Longleftrightarrow \quad g I h^\Psi \quad (\Longleftrightarrow h J g^\Phi).$$

Conversely, if R is a dual bond between (G, M, I) and (H, N, J) , then

$$\begin{aligned} g^\Phi &:= g^{RJ} & \text{for } g \in G, \\ h^\Psi &:= h^{RI} & \text{for } h \in H \end{aligned}$$

defines a relational Galois connection. The two constructions are inverse to each other⁶.

Proof. If R is defined as above from a relational Galois connection (Φ, Ψ) we get

$$g^R = g^{\Phi J} \quad \text{and} \quad h^R = h^{\Psi I} \quad \text{for } g \in G, h \in H.$$

Then obviously R is a dual bond. Conversely, if R is a dual bond, then $g^R = g^{RJJ}$ and thus

$$h J g^\Phi \Longleftrightarrow h \in g^{\Phi J} = g^{RJJ} = g^R \Longleftrightarrow g R h,$$

and analogously $g I h^\Psi \Longleftrightarrow g R h$. This shows that (Φ, Ψ) is a relational Galois connection. \square

As a corollary, we obtain a generalised version of Birkhoff's construction:

Corollary 1 ([5], Theorem 53). For every dual bond $R \subseteq G \times H$,

$$\phi(X, X^I) := (X^R, X^{RJ}), \quad \psi(Y, Y^J) := (Y^R, Y^{RI})$$

defines a Galois connection between $\underline{\mathfrak{B}}(G, M, I)$ and $\underline{\mathfrak{B}}(H, N, J)$. Conversely, for every Galois connection (ϕ, ψ) ,

$$R := \{(g, h) \mid (g^{II}, g^I) \leq \psi(h^{JJ}, h^J)\} = \{(g, h) \mid (h^{JJ}, h^J) \leq \phi(g^{II}, g^I)\}$$

is a dual bond. The two constructions are inverse to each other.

Theorem 2 has several nice consequences. First of all, it shows that relational Galois connections lead to closure operators:

⁵ Called *biclosed relation* by Domenach and Leclerc [3].

⁶ And induce a *dual* isomorphism between the lattice of all relational Galois connections and the lattice of all dual bonds.