Daylight Illumination-Color-Contrast Tables

for full-form objects

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PREFACE

This book is the outcome of a major computational project concerning the illumination, color, and contrast conditions in naturally illuminated objects, which has been pursued for several years by the Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt (DFVLR), Oberpfaffenhofen, West Germany, in cooperation with the Meteorologisches Institut der Universität München (MIM), Munich, West Germany.

The purpose of the project was to establish a comprehensive set of tables, from which often-needed reference and engineering data can be taken on the inherent illumination, color, and contrast in a generalized full-form model object illuminated by the sun, the sky, and light reflected from the ground, and viewed by a nearby observer; all for the case of a representative, turbid but cloudless continental atmosphere stringently defined in terms of physics, for a variety of solar elevations, and for a multitude of observation situations.

The computations leading to the tables pertaining to the luminance and color distributions in the sky were carried out at the MIM under the supervision of H. Quenzel, who is also the originator of the iterative computational programs involved, with the assistance of R. Wendling. The computations regarding the conditions in the object (irradiation, illumination, color, contrast) were made at the computation center of the DFVLR by W. Kweta, who also took care of the transcribing and editing of the data furnished by the MIM. M. R. Nagel originated and directed the overall effort and also formulated the mathematics for the object-related part of the tabulations.

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CONTENTS

0 INTRODUCTION

0.1 GENERAL

0.1.1 Background

The project from which this book was derived was originally conceived in support of the various remote sensing and image processing activities of the DFVLR, in particular, those depending on the quantitative photometric and colorimetric evaluation of photographs and other environmental records. The very nature of the resulting tables, however, suggests their applicability in a broad spectrum of technical and scientific disciplines concerned with problems of current interest in areas such as

(1) military search, reconnaissance, and camouflage tactics and technology (e.g., target detection),
(2) automatized land-use mapping (photometric ground feature identification),
(3) traffic safety and accident investigation (visibility of traffic hazards, obstacles, traffic signs, cablemarkers, etc.),
(4) vision-related architectural problems (natural illumination in offices, schools, auditoriums, museums, and so on),
(5) solar energy research,
(6) biological and balneological light climatology, meteorological visibility (contrast considerations), and textile colorimetry.

No similar, sufficiently detailed tables seem to exist in the open literature, which while built around strict principles of radiative transfer, emphasizes object-related rather than atmospheric–optical aspects. It appears useful and appropriate, therefore, to make a selection of the computed tables available to the scientific–technological community. The present time seems particularly suitable for such a project, because the space program has produced new and well-founded basic information, because new international conventions have recently been concluded on the fundamental physical constants, and also because new highly efficient computation software suitable for this purpose has become available. This combination of new knowledge and potential makes it possible to synthesize, finally, the entire sun–atmosphere–ground–object–observer system in a manner sensible from the viewpoints of physics and mathematics as well as from that of practical application.

In preparing this volume, the authors recognize that, in an era of computerized data banks, any hard-copy tabulation may appear out-of-step to some researchers—though only to the rather few in rather well-provided-for institutions. The present volume directs itself, first, to the large majority of lesser-privileged users all over the world who are left without ready access to a computer containing the required data; second, to those who simply prefer a book within an arm's reach to a computer terminal even as near as a short walk down the corridor; and third, to those just in search of another set of well-defined data.

Examples of earlier work in this area are by Wiener (1900), Kimball and Hand (1922), Elvegard and Sjöstedt (1940), Gordon (1964), and Valko (1970); others are given by Nagel *et al.* (1974). Many of the previous publications concern themselves only with single aspects of object illumination—illuminance, color, or contrast—and usually consider but few surface orientations in the object, or in the case of contrast, only a small number of viewing directions.

The two best-known of the very few tabulations concerning the atmospheric–optical part of the sun–observer system mentioned—both of them computer-calculated—are those by Coulson *et al.* (1960), which considers only a pure, though multiscatter, Rayleigh atmosphere, and by de Bary *et al.* (1965), which allows for a turbid atmosphere, but for single scattering only. Finally, one must mention in this connection the empirical formulas for the (relative) luminance distribution in the sky, which have more recently been adopted by the Commission Internationale de l'Éclairage (CIE, 1973). No specific turbidity condition can be associated with the values obtained through these formulas, however, since they in essence represent an average of a great number of measurements under a wide variety of turbid clear-sky situations and because all such empirical data—this applies also to those from still ongoing programs, e.g., Duntley *et al.* (1972)—are susceptible to the influence of many time-variable, local, environmental, and experimental factors.

Of necessity, all this leads to mutual inconsistencies among the various authors and unsatisfactory coverage of conditions, and it excludes the possibility of considering problems of optimalization, minimum-maximum properties, and probability that are so important, for example, in studies relating to target detection, equipment design, and observational tactics.

For the same reason, the earlier publications hardly permit reasonable estimates of the variations in the illuminance, color, or observable contrast in an object, when the photometry in its vicinity changes, when the object or the groundcover is colored, or when different observational tactics or instrumentation (viewing direction, field of view) are used.

The DFVLR-MIM concept attempts to answer to such needs,

(1) by starting out from no more experimental data and assumptions than necessary to clearly define a basic environment–optical situation, and then applying to them known laws of physics and mathematics to derive the irradiation and illumination conditions under a clear sky, for a number of solar elevations;

(2) by using an easily interpretable basic "full-form" model object—in this case a sphere—by tabulating the data for a narrow mesh of spatial coordinates having 325 reference

points in a half-sphere—enough to permit reasonably accurate interpolation—and by considering also a variety of deviations (colored ground, certain aspects of gloss, etc.) from that basic model;

(3) by assuming an even narrower mesh of spatial coordinates—2160 points in the half-sky—in the model atmosphere illuminating the object model;

(4) by assuming an observer viewing the object from many directions within a full 4π observation space around the object with either an unobstructed and a limited field of view; and

(5) by making the data presented fully traceable, in terms of physics, through a presentation progressing step by step from the basic irradiation data to the object-related tables regarding the illuminance, color, and contrast in the sphere.

While such scientific considerations of course determine the organization of the book as a whole, others require holding it to a reasonable size. For example, a tabulation of the spectral radiance distributions in the sky, if carried out in a detail comparable with that for sky luminance, would by itself fill an entire volume such as the present book. Therefore, where a choice had to be made, it was usually to emphasize aspects of visual sensing (luminance, illuminance, color, visual contrast) in preference to those of radiometry (radiance, irradiance, spectrum), and coverage of conditions for the medium-latitude regions of Earth is given preference before that for conditions in others. A description of the model used, and the mathematical relationships involved are given in the other sections of this introduction, along with the necessary instructions for the user.

The computer used by the MIM was the IBM 360/91–370/145 of the Max-Planck-Institut für Plasmaphysik at Garching, near Munich; the calculations made at the DFVLR were done on their Telefunken TR 440.

0.1.2 Description of the Model

0.1.2.1 General

The basic model assumed for the project and covered in the fully executed tables is supposed to simulate a simple outdoors scene consisting of a three-dimensional "full-form" object—in this case a white sphere—suspended in the air at short distance above the ground and illuminated by the sun, by a moderately turbid, cloudless sky, and by the light reflected from the ground. (In a full-form object, all points of the surface are exposed to the radiation emanating from within a full 2π sr solid angle of the object's environment.) This scene is viewed by a nearby human observer—in some cases using an instrument with a limited field of view—or by a sensor, respectively, whose spectral sensitivity matches that of the human photopic eye.

By means of several auxiliary tables and supplementary calculations it is also possible to consider certain aspects of a model scene modified, for example, with regard to reflectance or color in the object and the ground cover, or with regard to a specularly reflecting object surface. Sensibly applied deviations from the geometry or composition of the basic model scene are permissible.

0.1.2.2 The Sun

The basic solar spectral irradiances at the top of the terrestrial atmosphere used in the tables are those published by Thekaekara (1971), which are generally considered the most reliable ones available. The sun is assumed to be at its medium distance from Earth, having an apparent radius of the luminous disk of 16 minutes of arc. The tables cover conditions for seven solar elevations: 5, 10, 20, 30, 45, 60, and 90° above the horizon. This selection will make graphic interpolations of most illumination data possible and should cover adequately the conditions in all geographic regions on Earth. The 90° tables will be helpful in theoretical studies; the 5° elevation is considered the lowest permissible altitude in connection with the plane-parallel atmosphere adopted for the model.

0.1.2.3 Atmosphere

The atmosphere as a whole is taken to be plane-parallel with constant vertical mixing ratios for all components, except that all ozone is concentrated in its uppermost layer. An amount of 0.35 cm NTP of O_3 is assumed. This represents a reasonable global annual average for continental midlatitudes (Gebhart *et al.*, 1970). The resulting subozonal spectral irradiances for perpendicular incidence through a vertical column of the atmosphere are listed in the Basic Data tables (Table 1.00.0.1) along with the basic spectral solar irradiances and the spectral ozone absorption coefficients, the latter of which are by Vigroux (1953).

The aerosol particle size distribution used here for an average for continental aerosols is the pure power law distribution

$$dN/dr \sim r^{-(\nu^*+1)}$$

where dN is the number of particles having radii within the range from r to $r + dr$ and the values used for the parameters are

$$r_{\min} = 0.04 \quad \mu\text{m}, \qquad r_{\max} = 20 \quad \mu\text{m}, \qquad \nu^* = 3.3$$

This size distribution is very similar to the "haze C" distribution reported by Deirmendjian (1964).

The spectral extinction coefficient caused by such aerosol particles results in a wavelength dependence

$$a_{\text{Aer},\lambda} \sim \lambda^{-1.3}$$

Although it is known that the wavelength exponent actually varies considerably (Volz, 1956), the value 1.3 is a widely accepted average of measured values as stated by Ångström (1961).

As the mean value of the complex refractive index for the continental aerosol particles,

$$m = 1.50 - 0.02i$$

has been taken. This value is widely accepted in the scientific community and has frequently been measured (e.g., Hänel, 1976; Fischer, 1973; Lin *et al.*, 1973). The imaginary part of the refractive index causes a slight absorption fraction (= 1 − "albedo of single scattering") of the aerosol particles. The size distribution of the aerosol particles, together with their refractive index, further determines the normalized scattering function $p'_{\text{Aer}}(\Psi)$ (see Section 0.2.1.9), which is a part of the normalized scattering function $p'(\Psi)$ for the mixture of air molecules and aerosol particles.

In all computations, a value

$$p = 1013.25 \text{ mb}$$

is used for the sea level atmospheric air pressure.

Together with this air pressure, the amount of aerosol particles is given by the subozonal spectral turbidity factor TR_λ (for a definition see Section 0.2.1.15). At wavelength 550 nm TR_λ has been taken to be

$$TR_{550} = 3.21$$

This value, in turn, results from an Ångström turbidity coefficient $\beta_{\text{Å}} = 0.1$, found to be a mean value for the widely varying amount of aerosol particles in the atmosphere (Valko, 1971; Nagel, 1975).

The values for the various basic attenuation data for the atmospheric model are listed in the Basic Data tables (See Table 1.00.0.2). The numerical values for the normalized scattering functions are also tabulated there (see Table 1.00.0.3). Some of them are shown in Fig.0.2–3.

0.1.2.4 Object

The object sphere is small and normally has an ideally diffusing, white surface, so that in general

$$\text{object reflectance } \rho O = 1$$

Conditions for colored objects can be derived by follow-up calculations, for which auxiliary irradiation tables are provided. A special table is included to make consideration of the specular solar reflection in a smooth glossy object surface possible. The basic refractive index assumed for the surface layer is $nO = 1.5$, corresponding about to the center of the range of refractive indices for common glasses and for most paint vehicles. Using graphs also provided, other refractive indices can also be taken into account.

The following considerations were important in adopting this concept:

(1) The spherical shape of the object is easily accessible mathematically, and locations on the sphere can be readily interpreted in terms of surface orientations. This fact facilitates the use of the model in optimalization, maximum–minimum, and probability problems.

(2) Usually, the convex portions of an object surface are also the characteristic ones, because they determine the photometric, spectroscopic, and colorimetric appearance of the object and thereby its detectibility, classification, etc.

(3) The sphere must be small so that its shadow cannot materially affect the luminance distribution in the vicinity and thereby, conversely, on the sphere itself. Also, the model observer is supposed to be able to see a full half of the sphere from a short distance. This is possible only when the sphere is small. Smallness of the sphere is therefore a precondition for observing truly inherent photometric properties in the overall model.

(4) The assumption of a white or neutral gray model surface permits convenient computation of the individual components of illuminance, environment-induced color, and contrast, and subsequent application of the results to colored objects.

0.1.2.5 Ground Radiometric Properties

The ground is assumed to have a plane, neutral gray, perfectly diffusing surface extending into infinity in all directions. For the basic case,

$$\text{ground albedo } \rho = 0.1$$

has been assumed throughout the visible spectrum, but the conditions for other ground reflectances, as well as for colored ground covers, can also be calculated by means of tables contained in the book.

The assumption of a perfectly diffusing ground is admittedly quite arbitrary. Still, it represents a serviceable compromise, until a well-founded mean reflectance function becomes available that would account for the anisotropic reflectance properties of ground coverings on a continental scale.

In another aspect, it has been shown that the luminance of the sky is, to some degree, also a function of ground reflectance (Möller and Quenzel, 1972). It might be argued, therefore, that it is not permitted to assume in our model any variation in ground reflectances, without changing its entire photometric structure. In practice this is not so for two reasons. First, even in the rather severe case of an overcast sky, the sky luminance is coupled to the ground reflectance only by multiple scattering processes, so that a certain variation in ground albedo achieves a much smaller variation in the luminance of the sky. Second, in the case of an object near the ground, only a relatively small ground area in the immediate vicinity of the object contributes to the illumination conditions on its surface. If the ground reflectance in just that small area is changed, this will not perceptibly influence the average ground reflectance in the much larger area illuminating the sky, and the validity of the model will be maintained.

0.1.2.6 Observer and Observation Geometry

The model object is assumed to be viewed by an imaginary observer (or sensor). In some cases, this observer is supposed to survey a full half of the sphere; in others he is stationed so close to the object's surface that only a portion of the sphere is within his field of view. This latter situation may also occur when the observer uses an instrument with a narrow field of view.

The spectral sensitivities of the observer's eye or of the sensor are as that of the Commission Internationale de l'Éclairage 1931 photopic standard observer (CIE, 1970). The eye is supposed to be adapted to the color of the equal-energy spectrum ($xW = yW = zW = 1/3$). These stipulations accommodate the case of visual reconnaissance and can also be realized for many photoelectric sensors and films commonly used in aerial photography, by applying the corresponding corrective filters.

As explained in Section 0.1.2.4, the observer must always be stationed near the object, in absolute terms, so that no disturbing photometric influences are introduced by the atmospheric path between the object and the observer. Since the sphere is small in diameter, this condition can be met even when the observer is required to be able to survey a full half of the sphere.

In all cases involving viewing by an observer, the principal reference direction is from the observer to the center of the object sphere. The geometrical relationships for the case of the limited field of view are explained in Section 0.3.9.1.

0.1.2.7 Modification of the Basic Model

Depending on the permissible error, about which the decision must be left to the user, the described model may be modified regarding both photometry and geometry without an adverse effect on the applicability of the tables. Some of the possible photometric modifications are discussed in Section 0.3 in conjunction with the computations for the basic model. As to changes in geometry, for example, all data on illumination, color, and contrast can still be taken either directly from the tables or via simple computation using the tables, when the shape of the object is changed to other forms—rotational, box-type or irregular—as soon as each point of the surface in question is exposed to a 2π solid angle of an open environment. Also, the size of the object may in the application assume that of real objects, and obscurations of the sky by a wall, mountain, etc. along small sections of the horizon, to small elevations of perhaps 2°, or a reasonable depression of the horizon, sensible variations in the absolute altitude or distance from the ground, etc. will in many cases be allowed.

Changes in the original assumptions regarding the atmosphere, e.g., turbidity, parts of the model would usually affect the distribution of the basic irradiance and thereby also that of luminance and color over the object surface in a complex way. However, analysis indicates that in the important case of the total irradiance, and thereby total illuminance, on the horizontal plane, for medium and high solar elevations, appreciable changes in turbidity may result in errors that may be acceptable for many applications. The irradiances would in this case be higher at the lower turbidity. When the surface slopes, however, the errors become larger in relation to the differing contributions of sun, sky, and ground to the total irradiation, all of which are affected differently by turbidity.

0.2 COMPUTATION OF LUMINANCE DISTRIBUTION IN THE SKY

0.2.0 Remarks on the Notation

In Section 0.2, the symbols for the radiometric quantities have been adopted mainly from Chandrasekhar (1950), but the names of the radiometric and photometric quantities are in accordance with the conventions of the Commission Internationale de l'Éclairage (CIE, 1970). The symbols for the photometric quantities, too, are in agreement with the CIE recommendations.

0.2.1 Solution of the Equation of Radiative Transfer

0.2.1.1 Equation of Radiative Transfer

The radiation field at any point in the atmosphere is described and connected with all parameters that influence this radiation field by the equation of radiative transfer (Chandrasekhar, 1950). For a plane-parallel, horizontally homogenous, and infinite atmopshere, this equation is given by

$$\mu \frac{dI^{tot}(\tau, \mu, \varphi)}{d\tau} = I^{tot}(\tau, \mu, \varphi) - J^{tot}(\tau, \mu, \varphi) \tag{1}$$

where:

I^{tot} total spectral radiance perpendicular to the direction of propagation in $W \cdot m^{-2} \cdot \mu m^{-1} \cdot sr^{-1} = W \cdot cm^{-2} \cdot cm^{-1} \cdot sr^{-1}$. It can be expressed by the sum

$$I^{tot} = I^{dir} + I^{dif} + I^{em} \tag{2}$$

I^{dir}	spectral radiance of direct solar radiation
I^{dif}	spectral radiance of diffuse (scattered) solar radiation
I^{em}	spectral radiance emitted by atmospheric components
$\tau = \tau(p)$	optical depth of the atmosphere between the upper boundary and the height, with air pressure p
$T = \tau(p_0)$	total optical depth of the atmosphere between the upper boundary and the ground
p_0	air pressure at ground level
a_{Scat}	spectral scattering coefficient related to the total vertical column of the atmosphere
a_{Abs}	spectral absorption coefficient related to the total vertical column of the atmosphere
$a_{Ext} = a_{Scat} + a_{Abs}$	spectral extinction coefficient related to the total vertical column of the atmosphere; therefore,

$$\tau = \tau(p) = \int_0^p (a_{Scat} + a_{Abs}) \frac{1}{p_0}\, dp$$

$$T = \tau(p_0) = \int_0^{p_0} (a_{Scat} + a_{Abs}) \frac{1}{p_0}\, \mathrm{dp}$$

k	spectral absorption fraction, $= a_{Abs}/a_{Ext} = 1 - a_{Scat}/a_{Ext} = 1 -$ "albedo of single scattering"
μ, φ	coordinates of any point of the atmosphere or its boundary in the direction (μ, φ)
φ	azimuth of the point with zero azimuth always equal to sun's azimuth
μ	$=\cos z = \sin h$, with z = zenith angle and h = elevation angle
J^{tot}	total spectral source function, which can be expressed as the sum

$$J^{tot} = J^{dir} + J^{dif} + J^{em} \tag{3}$$

with

$$J(\tau, \mu, \varphi) = S(-\mu_0', \varphi_0') e^{-\tau/\mu_0} (1 - k) p'(\mu, \varphi; -\mu_0', \varphi_0') \tag{4}$$

$S(-\mu_0', \varphi_0')$	spectral irradiance (flux density) at the top of the atmosphere due to the extraterrestrial solar radiation perpendicular to the direction of propagation in $W \cdot m^{-2} \cdot \mu m^{-1}$
$p'(\mu, \varphi; \mu', \varphi')$	spectral normalized scattering function in sr^{-1} with normalization $\int_{sphere} p'(\mu, \varphi; \mu', \varphi')\, d\omega = 1$, gives the relative portion of the radiation scattered into direction (μ, φ), if the scattering volume is irradiated from direction (μ', φ')
$p(\mu, \varphi; \mu', \varphi')$	spectral scattering function defined by $a_{scat}\, p'(\mu, \varphi; \mu', \varphi')$

$$J^{dif}(\tau, \mu, \varphi) = (1 - k) \int_0^{2\pi} \int_{-1}^{+1} p'(\mu, \varphi; \mu', \varphi') I^{dif}(\mu', \varphi')\, d\mu'\, d\varphi' \tag{5}$$

$$J^{em} = kB(T) \tag{6}$$

$B(T)$	Planck function at absolute temperature T

In the visible and near infrared region of the spectrum, the values of J^{em} are negligibly small in comparison with the values of J^{dir} and J^{dif} and will be ignored from now on. The same holds for I^{em}.

Considering the law of extinction (Bouguer–Lambert law)

$$\mu \, dI^{\mathrm{dir}}/d\tau = I^{\mathrm{dir}} \tag{7}$$

and using expressions (2) and (3), Eq. (1) can be written

$$\boxed{\underbrace{\mu \, dI^{\mathrm{dif}}/d\tau}_{\text{term 0}} = \underbrace{I^{\mathrm{dif}}}_{\text{term 1}} - \underbrace{J^{\mathrm{dif}}}_{\text{term 2}} - \underbrace{J^{\mathrm{dir}}}_{\text{term 3}}} \tag{8}$$

where J^{dir} and J^{dif} are expressed by Eqs. (4) and (5), respectively.

The physical meaning of Eq. (8) is as follows: A variation of the field of diffuse radiation (term 0) results from the difference between a decrease (at increasing τ) due to extinction of scattered radiation (term 1) and two terms of increase that are due to more than one scattering process of each photon, i.e., multiple scattering (term 2), and only one scattering process of each photon, i.e., primary scattering (term 3).

The integrodifferential equation (8) has no analytical solution. The iterative method used here, the method of successive scattering, is based on Feigelson *et al.* (1960) and Korb and Möller (1962). Earlier work on this field has been done by Lenoble (1954).

0.2.1.2 Formal Solution

The formal solutions of Eq. (8) for the up-welling radiances $I^{\mathrm{dif}}(\tau, +\mu, \varphi)$ and the down-welling radiances $I^{\mathrm{dif}}(\tau, -\mu, \varphi)$ are

$$\begin{aligned} I^{\mathrm{dif}}(\tau, +\mu, \varphi) &= I^{\mathrm{dif}}(T, \mu, \varphi) e^{-(T-\tau)/\mu} \\ &\quad + \int_{\tau}^{T} (J^{\mathrm{dif}}(t, +\mu, \varphi) + J^{\mathrm{dir}}(t, +\mu, \varphi) e^{-(t-\tau)/\mu} \frac{dt}{\mu} \\ &\quad 0 < \mu \leq 1 \end{aligned} \tag{9}$$

$$\begin{aligned} I^{\mathrm{dif}}(\tau, -\mu, \varphi) &= I^{\mathrm{dif}}(0, -\mu, \varphi) e^{-\tau/\mu} \\ &\quad + \int_{0}^{\tau} (J^{\mathrm{dif}}(t, -\mu, \varphi) + J^{\mathrm{dir}}(t, -\mu, \varphi) e^{-(\tau-t)/\mu} \frac{dt}{\mu}, \\ &\quad 0 < \mu \leq 1 \end{aligned} \tag{10}$$

with J^{dir} and J^{dif} according to Eqs. (4) and (5), respectively.

The upper-boundary condition is

$$\boxed{I^{\mathrm{dif}}(0, -\mu, \varphi) = 0} \tag{11}$$

which means there is no incoming diffuse radiation from outside the atmosphere. The incoming radiation consists of the direct solar radiation only.

The lower-boundary condition is

$$\boxed{\begin{aligned} I^{\mathrm{dif}}(T, +\mu, \varphi) = &\int_{0}^{2\pi} \int_{0}^{1} \gamma_{\mathrm{r}}(\mu, \varphi; -\mu', \varphi') I^{\mathrm{dif}}(T, -\mu', \varphi') \mu' \, d\mu' \, d\varphi' \\ &+ \gamma_{\mathrm{r}}(\mu, \varphi, -\mu_0', \varphi_0') S(-\mu_0', \varphi_0') \mu_0 e^{-T/\mu_0} \end{aligned}} \tag{12}$$

where $\gamma_{\mathrm{r}}(\mu, \varphi; \mu', \varphi')$ is the spectral bidirectional reflectance distribution function in sr^{-1}. The lower-boundary condition determines the reflected radiation field due to the reflection properties of the ground and to the diffuse and direct irradiation.

0.2.1.3 *Formal Solution in Terms of Coefficients of Spherical Harmonics and Fourier Series*

For a numerical iterative solution of the equation of radiative transfer, a method based on the expansion of the scattering function onto a series of Legendre polynomials is used.

The scattering function $p(\mu, \varphi; \mu', \varphi')$ is not really a function of four independent variables $\mu, \varphi; \mu', \varphi'$ but depends on the scattering angle ψ, which can be expressed by

$$\cos\psi = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos(\varphi' - \varphi) \quad (13)$$

$p(\psi)$ can be expressed by

$$p(\psi) = \sum_{l=0}^{N} p_l P_l(\cos\psi) \quad (14)$$

where p_l are the coefficients of the expansion of the scattering function $p(\psi)$ into a series of Legendre polynomials $P_l(\cos\psi)$, which are spherical harmonics of the first kind.

The p_l are determined as usual by

$$p_l = (l + \tfrac{1}{2}) \int_{-1}^{+1} p(\psi) P_l(\cos\psi)\, d\cos\psi \quad (15)$$

Combining Eqs. (14) and (13) yields

$$p(\mu, \varphi; \mu', \varphi') = \sum_{m=0}^{N} (2 - \delta_{0,m}) \left\{ \sum_{l=m}^{N} p_l^m P_l^m(\mu) P_l^m(\mu') \right\} \cos m(\varphi' - \varphi) \quad (16)$$

for $0 \leq m < N$ and $l = m, \ldots, N$, where $P_l^m(\mu)$ are the associated Legendre polynomials,

$$p_l^m = p_l \frac{(l - m)!}{(l + m)!} \quad (17)$$

and

$$\delta_{0,m} = \begin{cases} 1 & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m \neq 0 \end{cases} \quad (18)$$

The $P_l^m(\mu)$ decrease very rapidly with increasing l and m. For instance, P_{28}^{28} is smaller than 10^{-78}. On the other hand, $(l + m)!$ increases very rapidly. This behavior is unsuitable for use by a computer. Therefore, renormalized associated Legendre polynomials are used as

$$Y_l^m(\mu) = [(l - m)!/(l + m)!]^{1/2} P_l^m(\mu) \quad (19)$$

Now the scattering function can be expressed by

$$\boxed{p(\mu, \varphi; \mu', \varphi') = \sum_{m=0}^{N} (2 - \delta_{0,m}) \left\{ \sum_{l=m}^{N} p_l Y_l^m(\mu) Y_l^m(\mu') \right\} \cos m(\varphi' - \varphi)} \quad (20)$$

Furthermore, the radiance $I^{\text{dif}}(\tau, \mu, \varphi)$ is expanded into a Fourier series:

$$I^{\text{dif}}(\tau, \mu, \varphi) = \sum_{m=0}^{N} \{I^{(m)}(\tau, \mu) \cos m\varphi + I^{*(m)}(\tau, \mu) \sin m(-\varphi)\} \quad (21)$$