Graduate Texts in Mathematics

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Differential Geometry

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Differential Geometry: Manifolds, Curves, and Surfaces

Springer-Verlag

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Marcel Berger Bernard Gostiaux

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Preface

This book consists of two parts, different in form but similar in spirit. The first, which comprises chapters 0 through 9, is a revised and somewhat enlarged version of the 1972 book *Géométrie Différentielle*. The second part, chapters 10 and 11, is an attempt to remedy the notorious absence in the original book of any treatment of surfaces in three-space, an omission all the more unforgivable in that surfaces are some of the most common geometrical objects, not only in mathematics but in many branches of physics.

Géométrie Différentielle was based on a course I taught in Paris in 1969–70 and again in 1970–71. In designing this course I was decisively influenced by a conversation with Serge Lang, and I let myself be guided by three general ideas. First, to avoid making the statement and proof of Stokes' formula the climax of the course and running out of time before any of its applications could be discussed. Second, to illustrate each new notion with non-trivial examples, as soon as possible after its introduction. And finally, to familiarize geometry-oriented students with analysis and analysis-oriented students with geometry, at least in what concerns manifolds.

To achieve all of this in a reasonable amount of time, I had to leave out a detailed review of differential calculus. The reader of this book should have a good calculus background, including multivariable calculus and some knowledge of forms in $\mathbf{R}^n$ (corresponding to pages 1–85 of [Spi65], for example). A little integration theory also helps. For more details, see chapter 0, where all of the necessary notions and results from calculus, exterior algebra and integration theory have been collected for the reader's convenience.

I confess that, in choosing the contents and style of *Géométrie Différentielle*, I emphasized the esthetic side, trying to attract the reader with theorems that are natural and simple to state, instead of providing an exhaustive exposition of the fundamentals of differentiable manifolds. I also decided to include a larger number of global results, rather than giving detailed proofs of local results.

More specifically, here are some of the contents of chapters 1 through 9:

—We start with a somewhat detailed treatment of differential equations, not only because they are used in several parts of the book, but because they tend to be given less an less weight in the curriculum, at least in France.

—Submanifolds of $\mathbf{R}^n$, although sometimes included in calculus courses, are then presented in detail, to pave the way for abstract manifolds.

—Next we define abstract (differentiable) manifolds; they are the basic stuff of differential geometry, and everything else in the book is built on them.

—Five examples of manifolds are then given and resurface several times along the book, thus serving as unifying threads: spheres, real projective spaces, tori, tubular neighborhoods of submanifolds of $\mathbf{R}^n$, and one-dimensional manifolds, i.e., curves. Tubular neighborhoods and normal bundles, in particular, form a class of examples whose study is non-trivial and illustrates a number of more or less refined techniques (chapters 2, 6, 7 and 9).

—Several important topics, for example, Morse theory and the classification of compact surfaces, are discussed without proofs. These "cultural digressions" are meant to give the reader a more complete picture of differential geometry and how it relates with other subjects.

—Two chapters are devoted to curves; this is, in my opinion, justified, because curves are the simplest of manifolds and the ones for which we have the most complete results.

—The exercises consist of fairly concrete examples, except for a few that ask the reader to prove an easy result stated in the text. They range from very easy to very difficult. They are in large measure original, or at least have not appeared in French books. To tackle the more difficult exercises the reader can refer to [Spi79, vol. I] or [Die69].

* * *

In deciding to add to the original book a treatment of surfaces, I faced a dilemma: if I were to maintain the leisurely style of the first nine chapters, I would have to limit myself to the basics or make the book far too long. This is especially true because one cannot talk about surfaces in depth without distinguishing between their intrinsic and extrinsic geometries. Once again the desire to give the reader a global view prevailed, and the solution I chose was to be much more terse and write only a kind of "travel guide," or extended cultural digression, omitting details and proofs. Given the abundance of good works on surfaces (see the introduction to chapter 10) and the great number of references sprinkled throughout our material, I feel that the interested reader will have no difficulty in filling in the picture.

Chapter 10, then, covers the local theory of surfaces in $\mathbf{R}^3$, both intrinsic (the metric) and extrinsic (the embedding in space). The intrinsic geometry of surfaces, of course, is the simplest manifestation of riemannian geometry, but I have resisted the temptation to talk about riemannian geometry in higher dimension, even though the field has witnessed spectacular advances in recent years.

Chapter 11 covers global properties of surfaces. In particular, we discuss the Gauss–Bonnet formula, surfaces of constant or bounded curvature, closed geodesics and the cut locus (part I, intrinsic questions); minimal surfaces, surfaces of constant mean curvature and Weingarten surfaces (part II, extrinsic questions).

* * *

The contents of this book can serve as a basis for several different courses: a one-year junior- or senior-level course, a one-semester honors course with emphasis on forms, a survey course on surfaces, or yet an elementary course emphasizing chapters 8 and 9 on curves, which can stand more or less on their own, together with section 7.6.

The reader who wants to go beyond the contents of this book will find a number of references inside, especially in chapters 10 and 11, but here are some general ones: [Mil63] is elementary, but a pleasure to read, as is [Mil69], which covers not only Morse theory but many deep applications to differential geometry; [Die69], [Ste64], [Hic65] and [Hu69] cover much of the same ground as as this book, with differences in emphasis; [War71] has a good treatment of Lie groups, which are only mentioned in this work; [Spi79], whose first volume largely overlaps with our chapters 1 to 9, goes on for four more and is especially lucid in offering different approaches to riemannian geometry and expounding its historical development; and [KN69] is the ultimate reference work.

I would like to thank Serge Lang for help in planning the contents of chapters 0 to 9, the students and teaching assistants of the 1969–1970 and 1970–1971 courses for their criticism, corrections and suggestions, F. Jabœuf for writing up sections 7.7 and 9.8, J. Lafontaine for writing up numerous exercises and for the proof of the lemma in 9.5. For feedback on the two new chapters I'm indebted to thank D. Bacry, J.-P. Bourguignon, J. Lafontaine and J. Ferrand.

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Marcel Berger
I.H.E.S, 1987

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CHAPTER 0

Background

This chapter contains fundamental results from exterior algebra, differential calculus and integration theory that will be used in the sequel. The statements of these results have been collected here so that the reader won't have to hunt for them in other books. Proofs are generally omitted; the reader is referred to [Car71], [Dix68] or [Gui69].

0.0. Notation and Recap

0.0.1. Notation

0.0.2. Let X be a topological space. We denote by $O(X)$ the set of open subsets of X; by $O_x(X)$ the set of open subsets of X containing a point $x \in X$; and by $O_A(X)$ the set of open subsets of X containing a subset $A \subset X$.

0.0.3. If X is a metric space, we let $B(a,r)$ and $\overline{B}(a,r)$ be the open and closed balls of radius r and center a. When $X = \mathbf{R}^d$ we write $B_d(0,1)$ instead of $B(0,1)$.

0.0.4. If E and F are vector spaces over the same field, we let $L(E;F)$ be the vector space of continuous linear maps from E into F (if E and F have finite dimension every linear map is continuous). If $F = \mathbf{R}$ we write E^* instead of $L(E;\mathbf{R})$; this space is called the *dual* of E and its elements are continuous *linear forms* on E.

0.0.5. If X and Y are topological spaces we let $C^0(X;Y)$ be the set of continuous maps from X into Y.

0.0.6. The algebra of continuous functions from X into $\mathbf{R}$ is denoted by $C^0(X)$.

0.0.7. Recap

0.0.8. If X is a compact topological space, $C^0(X)$, with the norm of uniform convergence, is a complete topological space [Car71, I.1.2, example 2].

0.0.9. A finite-dimensional vector or affine space over $\mathbf{R}$ has a canonical topology, given by a norm. All norms are equivalent; in particular, we can take any Euclidean norm [Car71, I.1.6.2].

0.0.10. Example. If E and F are finite-dimensional vector spaces, so is $L(E;F)$: its dimension is equal to $\dim(E) \cdot \dim(F)$.

If E and F are normed vector spaces, $L(E;F)$ has a canonical norm, defined by

$$\|f\| = \sup\{\|f(x)\| : \|x\| = 1\}.$$

Then $\|f \circ g\| \le \|f\| \cdot \|g\|$ [Car71, equation I.1.5.1], and $L(E;F)$ is a Banach space if F is [Car71, theorem I.1.4.2].

0.0.11. If E and F are isomorphic vector spaces, denote by $\mathrm{Isom}(E;F)$ the set of isomorphisms from E to F. Then

0.0.12 $$\phi : \mathrm{Isom}(E;F) \ni f \mapsto f^{-1} \in \mathrm{Isom}(F;E)$$

is continuous for the norm defined in 0.0.10, as the reader should check [Car71, theorem I.1.7.3].

0.0.13. Lipschitz and contracting maps [Car71, I.4.4.1]

0.0.13.1. Definition. Let X and Y be metric spaces. A map $f : X \to Y$ is a *k-Lipschitz map* if there exists $k \in \mathbf{R}$ such that

$$d\big(f(x), f(y)\big) \leq k\, d(x,y)$$

for every $x, y \in X$.

A map $f : X \to Y$ is *locally Lipschitz* if for every $x \in X$ there exists $V \in O_x(X)$ such that $f|_V$ is Lipschitz. A map $f : X \to Y$ is *contracting* if it is k-Lipschitz with $k < 1$.

0.0.13.2. Theorem. *If X is a complete metric space and $t : X \to X$ is contracting, t has a unique fixed point, that is, there exists a unique z such that $t(z) = z$. In addition, $z = \lim_{n\to\infty} t^n(x)$ for every $x \in X$.* □

0.1. Exterior Algebra

Let E be a vector space and $E^* = L(E;\mathbf{R})$ its dual.

0.1.1. We denote by $\Lambda^r E^*$ the vector space of *alternating r-linear forms* on E, that is, continuous maps $\alpha : E^r \to \mathbf{R}$ linear in each variable and satisfying

$$\alpha(\ldots, x_i, \ldots, x_j, \ldots) = -\alpha(\ldots, x_j, \ldots, x_i, \ldots)$$

for every $1 \leq i \leq j \leq r$. One has $\Lambda^1 E^* = E^*$; by convention, $\Lambda^0 E^* = \mathbf{R}$. If E is n-dimensional, $\Lambda^r E^*$ has dimension $\binom{n}{r}$ if $r \leq n$ and dimension 0 if $r > n$ [Dix68, 37.1.11].

Recall that, if $f_1, \ldots, f_r$ are linear forms on E, we define $f_1 \wedge \cdots \wedge f_r \in \Lambda^r E^*$ by

0.1.2 $$(f_1 \wedge \cdots \wedge f_r)(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} \varepsilon_\sigma f_1(x_{\sigma(1)}) \ldots f_r(x_{\sigma(r)}),$$

where S_r is the symmetric group on r elements and $\varepsilon_\sigma = \pm 1$ depending on whether σ is an even or odd permutation.

0.1.3. Basis for $\Lambda^r E^*$. Let $\{e_1, \ldots, e_n\}$ be a basis for E and $\{e_1^*, \ldots, e_n^*\}$ the dual basis for E^*. Let $I = (i_1, \ldots, i_r)$ be an r-tuple such that

$$1 \le i_1 < i_2 < \cdots < i_r \le n.$$

The forms $e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_r}^*$, as I ranges over all such n-tuples, form a basis for $\Lambda^r E^*$ [Dix68, 37.1.9].

0.1.4. Exterior product of alternating forms. Consider $\alpha \in \Lambda^p E^*$ and $\beta \in \Lambda^q E^*$. The *exterior product* $\alpha \wedge \beta$, an alternating $(p+q)$-linear form, is defined as follows: let A be the subset of S_{p+q} consisting of permutations σ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \qquad \text{and} \qquad \sigma(p+1) < \cdots < \sigma(p+q).$$

Then

0.1.5
$$(\alpha \wedge \beta)(x_1, \ldots, x_{p+q}) = \sum_{\sigma \in A} \varepsilon_\sigma \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})$$

[Dix68, 37.2.5–11]. The exterior product is associative.

0.1.6. If $\alpha \in \Lambda^r E^*$, we say that r is the *degree* of α, and write $\deg \alpha = r$. If $\alpha \in \Lambda^r E^*$ and $\beta \in \Lambda^s E^*$ we have

0.1.7
$$\beta \wedge \alpha = (-1)^{\deg \alpha \deg \beta} \alpha \wedge \beta.$$

Thus the exterior product makes the vector space

$$\Lambda E^* = \bigoplus_{r=0}^{\dim E} \Lambda^r E^*$$

into an associative and anticommutative algebra.

0.1.8. Pullbacks. For $f \in L(E; F)$ we define $f^* \in L(\Lambda^r F^*; \Lambda^r E^*)$ by

0.1.9
$$f^* \beta(u_1, \ldots, u_r) = \beta\bigl(f(u_1), \ldots, f(u_r)\bigr)$$

for every $\beta \in \Lambda^r E^*$ and every $u_1, \ldots, u_r \in E$. One immediately sees that

0.1.10
$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta).$$

If $f \in L(E; F)$ and $g \in L(F; G)$ we have

0.1.11
$$(g \circ f)^* = f^* \circ g^*.$$

0.1.12. For $f \in L(E; E)$ and $\beta \in \Lambda^n E^*$, where n is the (finite) dimension of E, we have

0.1.12.1
$$f^* \beta = (\det f) \beta.$$