John W. Morgan

The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds

塞伯格－威顿方程及其在光滑四流形拓扑中的应用

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by

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Chapter 1

Introduction

Beginning with the groundbreaking work of Donaldson in about 1980 it became clear that gauge-theoretic invariants of principal bundles and connections were an important tool in the study of smooth four-dimensional manifolds. Donaldson showed the importance of the moduli space of anti-self-dual connections. The next fifteen years saw an explosion of work in this area leading to computations of Donaldson polynomial invariants for a wide class of four-dimensional manifolds, especially algebraic surfaces. These computations yielded many powerful topological consequences including, for example, the diffeomorphism classification of elliptic surfaces. For some of the results obtained by using these techniques see [1], [6], and [2].

Last fall, motivated by new work in quantum field theory, Seiberg and Witten [9], introduced a different gauge-theoretic invariant which they claimed should be closely related to Donaldson's invariants. Indeed they gave an explicit formula for the relationship of their invariant to Donaldson's. While the link claimed by Seiberg-Witten between their invariants and Donaldson's has not yet been established mathematically, it is clearly true and can be shown to hold in all computed examples. Nevertheless, one can forget this supposed link and work directly with the new invariants as a substitute for the anti-self-dual invariants. This has been the approach during the last year or so.

It was clear from the beginning that the new invariants would be easier to work with since they involved principal bundles with structure group the circle instead of the non-abelian groups such as $SU(2)$ which arise in Donaldson theory. The surprise was that such simple invariants could capture the subtlety that Donaldson's invariants revealed. But in short order, Witten [17] and then Taubes, Kronheimer, and Mrowka showed that indeed these new invariants did capture these subtleties and that they were easier to compute, at least in many cases. They did this by explicitly solving the Seiberg-Witten equations over Kähler surfaces. (See [3] for one account of the results for Kähler surfaces.) There followed in quick succession a series of remarkable theorems, each extending in a different way partial results from Donaldson's anti-self-dual theory. In fact, conjectures which seemed reasonable from the perspective of Donaldson theory but technically difficult, if not unreachable within that theory, suddenly became the standard test cases for the power of the new invariants. One by one these conjectures were established – leaving only one classical conjecture outstanding. The remaining one, called the $11/8ths$-Conjecture, deals with the quadratic forms that arise as intersection forms of simply connected spin four-manifolds.

It is the purpose of these notes to lay the groundwork for the Seiberg-Witten theory and then to show how one computes these invariants for most Kähler surfaces. We begin with the basics of Clifford algebras, spin structures and their cousins $Spin^c$-structures, spin representations, and spinor bundles. We then consider the Dirac operator on the spin bundles over a riemannian four-manifold. The connection with Kähler geometry is facilitated because of the close connection on a Kähler manifold between the Dirac operator and $\overline{\partial}$. All of this is examined in detail. The book of Lawson-Michelson [7] gives a complete introduction to this material as well as an elaborate treatment of many more topics in the spinor geometry.

We then exhibit the Seiberg-Witten equations and show how to use these equations to produce a finite dimensional manifold, the moduli space of solutions to these equations modulo changes of gauge, inside an infinite dimensional configuration space. The homology class of this moduli space is then the value of the Seiberg-Witten invariant. We establish analogues of familiar theorems from anti-self-dual theory. The equations defining the moduli space are elliptic modulo the group of changes of gauge. A generic perturbation of the equations leads to a smooth orientable moduli space whose dimension can be computed by the Atiyah-Singer index theorem. To orient this moduli space it suffices to choose an orientation of H^2_+ and H^1 of the underlying four-manifold. As long as $b_2^+ > 1$ the moduli space varies smoothly as we vary the metric and perturbation. When $b_2^+ = 1$ the generic perturbation yields a smooth moduli space, but there will be singularities when we vary the metric and perturbation. These singularities are reducible solutions to the equations. This leads to a chamber structure for the Seiberg-Witten invariants in case $b_2^+ = 1$, similar to what happens in the anti-self-dual theory.

We go on to establish one special property of this theory, namely the compactness of the moduli space of solutions. This is a consequence of a *priori* bounds for the pointwise norms of solutions to the equations. This result has no analogue for the anti-self-dual equations. Much of the geometric richness and much of the technical complexity of the anti-self-dual theory is directly related to the non-compactness of the moduli space. For both better and worse, that is all missing here. The Seiberg-Witten moduli spaces are compact and vary by a compact bordism as we vary the metric.

Having presented all these technical results, it is now clear that the homology class of the moduli space in the ambient space of configurations is an invariant of the underlying smooth four-manifold and the isomorphism class of the $Spin^c$ structure on that manifold. By definition, the Seiberg-Witten invariant of the $Spin^c$ structure is this homology class. We finish by explicitly computing the moduli spaces of solutions on 'most' algebraic surfaces, leading immediately to a computation of the Seiberg-Witten invariant for any $Spin^c$ structure over the surface. There are some special cases that we do not treat. These can, however, be treated by an elaboration of the techniques that we introduce. A complete discussion is contained in [3].

There is much more to be said about the Seiberg-Witten equations and the invariants that are defined from these equations. There are analogues of the theorems for Kähler manifolds which hold for symplectic manifolds, see [12, 13, 14]. There are also gluing theorems which lead to Meyer-Vietoris principals for solutions to the equations, see [8]. These then lead to product formulas for the invariants, which is one approach to gaining a more topological understanding of the invariants. In spite of the striking progress over the last year, much remains to be done. It is my hope that these notes will serve as an introduction making it possible for more mathematicians to contribute to this progress.

These notes are a written version of lecture series given at Columbia University and Princeton University during the past year. I wish to thank the graduate students at both Universities who attended these lectures and who, by their comments and questions, helped shape these notes.

Chapter 2

Clifford Algebras and Spin Groups

For any $n > 2$ the orthogonal group $SO(n)$ has fundamental group $\mathbf{Z}/2\mathbf{Z}$ and hence has a universal covering group called $Spin(n)$ which is a non-trivial double covering. There is a beautiful algebraic construction which yields the Spin groups (and much more as well). This is the subject of Clifford algebras.

2.1 The Clifford Algebras

An example. Before delving into the complexities of Clifford algebras in general, let us consider a simple example. Consider the unit sphere S^3 inside the quaternion algebra $\mathbf{H}$. Multiplication of quaternions induces a group structure on S^3. Let us consider the action of this group on $\mathbf{H}$ by conjugation

$$S^3 \times \mathbf{H} \to \mathbf{H}$$

$$(\alpha, \lambda) \mapsto \alpha\lambda\alpha^{-1}.$$

This action preserves the norm, i.e., is an orthogonal action. It also leaves invariant the center of $\mathbf{H}$ which is $\mathbf{R} \subset \mathbf{H}$, and hence it leaves invariant the perpendicular complement to $\mathbf{R}$ which is the three-dimenisonal space $\mathrm{Im}\,\mathbf{H}$ of purely imaginary quaternions. Of course, $\mathrm{Im}\,\mathbf{H}$ is naturally identified with the Lie algebra of S^3 and the action we are considering is the adjoint action of S^3 on its Lie algebra.

It is an easy geometric exercise to see that a unit quaternion α, different from ± 1, acts by conjugation on $\mathbf{H}$ preserving the complex plane $\mathbf{C}\alpha$ spanned by α and its perpendicular $\mathbf{C}\alpha j$. On the first complex plane conjugation by α acts trivially and on the second it acts by rotation through twice the angle θ between α and 1. It follows that the conjugation action of α on $\mathrm{Im}\,\mathrm{H}$ leaves invariant the line tangent to the circle generated by α and acts by rotation through 2θ on the perpendicular complement and thus that every rotation of $\mathbf{Im}\,\mathbf{H}$ is in the image of the representation

$$S^3 \to SO(\mathbf{Im}\,\mathbf{H}) = SO(3).$$

It is also clear that the kernel of this representation is the intersection of S^3 with the center of $\mathbf{H}$, which is the center of S^3 and is $\{\pm 1\}$.

In this way we construct the double covering group of $SO(3)$ and identify it with the group of quaternions of length one under multiplication.

The definition of the Clifford algebra associated to an positive-definite inner product space. Now let us turn to the general case. Let V be a finite dimensional vector space over $\mathbf{R}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$ leading to a norm denoted $|| \cdot ||$. We consider the tensor algebra

$$T(V) = \oplus_{n \geq 0} \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

generated by V. It is an associative algebra with unit 1. The Clifford algebra $Cl(V)$ generated by V is the quotient of $T(V)$ by the two-sided ideal generated by all elements of the form

$$v \otimes v + ||v||^2 1$$

for $v \in V$. Notice that the grading on $T(V)$ descends to a $\mathbf{Z}/2\mathbf{Z}$ grading on $Cl(V)$, giving a decomposition

$$Cl(V) = Cl_0(V) \oplus Cl_1(V)$$

where $Cl_0(V)$ is a subalgebra and $Cl_1(V)$ is a module over this subalgebra. Corresponding to this splitting, we can write any $v \in Cl(V)$ as $v_0 + v_1$. We denote by

$$\epsilon\colon Cl(V) \to Cl(V)$$

the algebra homomorphism that is multiplication by $+1$ on $Cl_0(V)$ and is multiplication by -1 on $Cl_1(V)$; i.e., $\epsilon(v_0 + v_1) = v_0 - v_1$.

Choosing an orthonormal basis $\{e_1, \ldots, e_n\}$ for V we can write $Cl(V)$ in terms of generators and relations. Namely, $Cl(V)$ is the algebra over $\mathbf{R}$ generated by $\{e_1, \ldots, e_n\}$ subject to the relations $e_i^2 = -1$ for all $i \leq n$ and $e_i \cdot e_j = -e_j \cdot e_i$ for all $i \neq j$. In particular, it follows that every element of $Cl(V)$ can be written uniquely as a sum of products of the form

$$e_{i_1} \cdots e_{i_t}$$

where $i_1 < \cdots < i_t$. Thus, the dimension of $Cl(V)$ as a real vector space is 2^d where d is the dimension of V over $\mathbf{R}$.

Examples. (i) Let $\mathbf{R}^n$ denote the usual Euclidean space of dimension n. Then $Cl(\mathbf{R}^1) = \mathbf{R}[x]/(x^2+1) \cong \mathbf{C}$. The subalgebra $Cl_0(\mathbf{R}^1)$ is identified with the reals and $Cl_1(\mathbf{R}^1)$ with the purely imaginary complex numbers.

(ii) Similarly, $Cl(\mathbf{R}^2)$ is the algebra generated by x, y subject to the relations

$$x^2 = -1; y^2 = -1; xy = -yx.$$

Hence, $Cl(\mathbf{R}^2)$ is isomorphic to the quaternion algebra $\mathbf{H}$. The subalgebra $Cl_0(\mathbf{R}^2)$ is generated by xy and can be identified with $\mathbf{C} \subset \mathbf{H}$.

(iii) $Cl(\mathbf{R}^3)$ is of dimension 8 over $\mathbf{R}$. It is generated by x, y, z with $x^2 = y^2 = z^2 = -1$ and $xy = -yx, xz = -zx, yz = -zy$. This algebra is isomorphic to $\mathbf{H} \oplus \mathbf{H}$. The isomorphism between $\mathbf{H}$ and the first factor, resp., the second factor, is given by sending $1, i, j, k$ to

$$\frac{1+xyz}{2}, \frac{xy-z}{2}, \frac{yz-x}{2}, \frac{zx-y}{2},$$

resp., to

$$\frac{1-xyz}{2}, \frac{xy+z}{2}, \frac{yz+x}{2}, \frac{zx+y}{2}.$$

The subalgebra $Cl_0(\mathbf{R}^3)$ is identified with the diagonal copy of $\mathbf{H}$ in this decomposition.

(iv) For any inner product space V, we have an isomorphism of algebras $Cl(V) \cong Cl_0(V \oplus \mathbf{R})$. Letting e be a unit vector in $\mathbf{R}$, the isomorphism is given by

$$v_0 + v_1 \mapsto v_0 + v_1 e.$$

It is an easy exercise to show that this map is an isomorphism of algebras. In particular, $Cl_0(\mathbf{R}^4)$ is isomorphic to $\mathbf{H} \oplus \mathbf{H}$.

Comparison with the Exterior Algebra. The grading on $T(V)$ induces an increasing filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$$

of $Cl(V)$ by linear subspaces; namely, we set $\mathcal{F}_t$ equal to the image in $Cl(V)$ of

$$\oplus_{n\leq t} \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}.$$

Clearly, the multiplication in $Cl(V)$ preserves this filtration in the sense that it induces maps

$$\mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{F}_{i+j}.$$

Thus, there is an associated graded algebra

$$\mathrm{Gr}_{\mathcal{F}_*}(Cl(V)) = \oplus_{n=0}^{\infty} \mathcal{F}_n / \mathcal{F}_{n-1}$$

with the induced multiplication.

Claim 2.1.1 $\mathrm{Gr}_{\mathcal{F}_*}(Cl(V))$ *is naturally isomorphic to the exterior algebra* $\Lambda^*(V)$.

Proof. Let $\widetilde{\mathcal{F}}_*$ be the increasing filtration of $T(V)$ coming from the grading. Clearly, there is a natural algebra isomorphism between $T(V)$ and $\mathrm{Gr}_{\widetilde{\mathcal{F}}_*}(T(V))$. Consequently, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{I}\left(v \otimes v + \|v\|^2 1\right) & \longrightarrow & T(V) & \longrightarrow & Cl(V) & \longrightarrow & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \\
\mathcal{I}(v \otimes v) & \longrightarrow & \mathrm{Gr}_{\widetilde{\mathcal{F}}_*}(T(V)) & \longrightarrow & \mathrm{Gr}_{\mathcal{F}_*}(Cl(V)) & \longrightarrow & 0.
\end{array}$$

Clearly, the quotient of $T(V)$ by the two-sided ideal generated by $v \otimes v$ for $v \in V$ is exactly the exterior algebra $\Lambda^*(V)$. Thus, this diagram induces the claimed isomorphism. □

There is a natural splitting σ of the map $Cl(V) \to \Lambda^*(V)$. The map σ is linear but not multiplicative. It is defined as follows. Consider an elementary element in $\Lambda^k V$ (; i.e., one contained in $\Lambda^k(W) \subset \Lambda^k(V)$ for some k-dimensional subspace $W \subset V$. Such an element can be written as $re_1 \wedge \cdots \wedge e_k$ where the e_i form an orthonormal basis for W and where $r > 0$. We define

$$\sigma(re_1 \wedge \cdots \wedge e_k) = re_1 \cdots e_k.$$

It is an easy exercise to show that this determines a well-defined mapping which splits the natural projection.

Using this isomorphism, we can view $Cl(V)$ as being given by a new multiplication on $\Lambda^*(V)$. This new multiplication is generated by

$$v \cdot (v_1 \wedge \cdots \wedge v_k) = v \wedge v_1 \wedge \cdots \wedge v_k - v \angle (v_1 \wedge \cdots \wedge v_k)$$

where $\angle$ is the contraction:

$$v \angle (v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle v, v_i \rangle v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k .$$

2.2 The groups $Pin(V)$ and $Spin(V)$

Let $Cl^\times(V)$ denote the multiplicative group of units of the algebra $Cl(V)$. We define the group $Pin(V)$ as the subgroup of $Cl^\times(V)$ generated by elements $v \in V$ with $\|v\|^2 = 1$. Notice that the given generators of $Pin(V)$ are units since the square of any one of them is -1. We let $Spin(V)$ be the intersection of $Pin(V)$ with $Cl_0(V)$, i.e., the kernel of the group homomorphism $Pin(V) \to \mathbf{Z}/2\mathbf{Z}$ induced by the splitting of $Cl(V) = Cl_0(V) \oplus Cl_1(V)$. Since the generators of $Pin(V)$ are contained in $Cl_1(V)$, $Spin(V)$ is the subgroup of index two consisting of all elements in $Pin(V)$ which can be written as a product on an even number of the given generators for $Pin(V)$. Let us compute these groups in the first three examples.

(i) The group $Pin(1)$ is the subgroup of $\mathbf{C}$ generated by $\pm i$. Hence it is a cyclic group of order 4. The subgroup $Spin(1)$ is the group of order two ± 1 inside $\mathbf{R}$.

(ii) The group $Pin(2)$ is the subgroup of $\mathbf{H}$ generated by the circle through j and k, i.e., by all elements of the form $\cos(\theta) j + \sin(\theta) k$ for $\theta \in S^1$. It is easy to see that this group is the union of two circles – the usual unit circle in the complex plane and j times it. Hence, the group $Spin(2)$ is isomorphic to S^1.

(iii) The group $Spin(3)$ is isomorphic to the group of unit quaternions in $Cl_0(\mathbf{R}^3) \cong \mathbf{H}$. To see this, notice that under the identification of $Cl(\mathbf{R}^3)$ with $\mathbf{H} \oplus \mathbf{H}$ given above, the vector space $\mathbf{R}^3$ is identified with all pairs $(\alpha, -\alpha)$ where α is a purely imaginary quaternion. It follows that for any pair α, β of purely imaginary unit quaternions the product

$$\alpha\beta \in Spin(V) \subset Cl_0(\mathbf{R}^3) = \mathbf{H} \stackrel{\Delta}{\hookrightarrow} \mathbf{H} \oplus \mathbf{H}.$$

It is easy to see that this set of products generates the group S^3 of all unit quaternions.

(iv) Let us consider $Spin(4) \subset Cl_0(\mathbf{R}^4) \cong \mathbf{H} \oplus \mathbf{H}$. Of course, under the identification $Cl(\mathbf{R}^3) \cong Cl_0(\mathbf{R}^4)$, the group $Spin(3)$ becomes a subgroup of $Spin(4)$. But there are many different three-dimensional subspaces of $\mathbf{R}^4$. For each such subspace we obtain an embedding of $Spin(3)$ into $Spin(4)$. It is an easy exercise to show that the union of these images generates all of $S^3 \times S^3 \subset \mathbf{H} \times \mathbf{H}$. This then identifies $Spin(4)$ with $SU(2) \times SU(2)$.

Notice that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, then every product $e_{i_1} \cdots e_{i_t}$ is an element of $Pin(V)$. This means that $Pin(V)$ contains a vector space basis for $Cl(V)$, and consequently that $Cl(V)$ is the smallest algebra over $\mathbf{R}$ containing $Pin(V)$ as a subgroup of its multiplicative group of units. Similarly, $Spin(V)$ contains an $\mathbf{R}$-basis for $Cl_0(V)$.

Corollary 2.2.1
- *Two (real or complex) representations of the algebra $Cl_0(V)$ whose restrictions to $Spin(V)$ are isomorphic representations are in fact isomorphic representations of the algebra.*
- *Let A be a (real or complex) module over $Cl_0(V)$ and let $A' \subset A$ be a subspace invariant under the induced action of $Spin(V)$. Then A' is a submodule for the $Cl_0(V)$ action.*

Proof. Suppose that two modules A and A' for $Cl_0(V)$ admit a linear isomorphism φ which commutes with the induced $Spin(V)$ actions. Then φ commutes with the actions of an $\mathbf{R}$-basis of $Cl_0(V)$ and hence commutes with the $Cl_0(V)$ actions. That is to say, φ is an isomorphism of $Cl_0(V)$-modules. This proves the first result. The second is established similarly. □

Notice that there are analogous results for $Cl(V)$ and $Pin(V)$.

Clearly, the natural action of the group $O(V)$ on V extends to an action of $O(V)$ on $Cl(V)$ as algebra automorphisms preserving the $\mathbf{Z}/2\mathbf{Z}$-grading. This action is effective and hence induces an embedding of $O(V)$ into the algebra automorphisms of $Cl(V)$. Since $Cl(V)$ is generated as an algebra by $V \subset Cl(V)$ and since $v \cdot v = -||v||^2 1$, it is easy to see that the image of this embedding consists of all the algebra automorphisms of $Cl(V)$ which preserve the subspace V. The subgroup $SO(V)$ is represented as the group of all algebra automorphisms of $Cl(V)$ which preserve V and act in an orientation-preserving fashion on it.

The group $Spin(V)$ acts on $Cl(V)$ via conjugation: $\sigma \cdot c = \sigma c \sigma^{-1}$. It is easy to see that this action also preserves the algebra structure and the