DIFFERENTIAL EQUATIONS

Ford

Differential Equations

LESTER R. FORD, Ph. D.

Professor Emeritus of Mathematics
Illinois Institute of Technology

SECOND EDITION

McGRAW-HILL BOOK COMPANY, INC.

New York Toronto London

1955

DIFFERENTIAL EQUATIONS

Library of Congress Catalog Card Number 54-12675

IX

21509

THE MAPLE PRESS COMPANY, YORK, PA.

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PREFACE

This second edition follows the use of the book as a text for more than twenty years, and this experience has guided the revision. Many parts have been rewritten and rearranged in the interests of a clearer presentation, a smoother and more natural approach, and a more teachable body of material. More exercises have been worked out as a guide to the student. Numerous additions to the lists of problems include many simple exercises as well as those which challenge the student's ability and insight. In response to a persistent demand a set of review exercises has been put at the end of Chap. 2. A complete set of answers has been included in the book.

The strictly new subject matter has, for the most part, been often used by the author as supplementary material: Riccati's equation, elastic vibrations, planetary motion, and, at the end of the book, the simple numerical methods which are used in the approximate solution of Laplace's equation. There have also been added several pages on the Laplace transform, designed to give the student some acquaintance with this popular tool.

LESTER R. FORD

A SHORT COURSE

The following sections are suggested for a one-semester course of four hours. With some variations this material has stood the test of long experience. The basic theory and the chief applications covered are: (1) an introduction to differential equations and various elementary applications; (2) the highly useful linear equations; (3) a classical equation in the complex domain; (4) an existence theorem; (5) the partial differential equation of the vibrating string; (6) a treatment of planetary motion.

For a still shorter course the instructor must sacrifice some of these materials. For a three-hour course the author suggests the omission of some of the starred items.

The book is rich in materials for special assignments to able students and in intriguing problems.

Chap. 1: 1–13, 15–17.
2: 1, 2*, 3*, 4, 7, 8.
3: 1–10, 13*.
4: 1, 2, 5–9.
5: 1, 2*–5*.
6: 4, 5, 11*, 12*.
7: 1, 2, 9.
11: 1*–4*, 6*.

CONTENTS

CHAPTER 4
GENERAL LINEAR EQUATIONS

Laplace Transforms

CHAPTER 5
THE METHOD OF SUCCESSIVE APPROXIMATIONS

CHAPTER 6
SYSTEMS OF ORDINARY EQUATIONS

CHAPTER 7
CERTAIN CLASSICAL EQUATIONS

CHAPTER 8
INTERPOLATION AND NUMERICAL INTEGRATION

Symbolic Methods

CHAPTER 9
THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

CHAPTER 10
PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

CHAPTER 11
PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

CHAPTER 1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

1-1. Differential Equations. An equation

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \tag{1-1}$$

involving a function $y(x)$ and certain of its derivatives is called a *differential equation.*

By the *order* of a differential equation is meant the order of the highest derivative which appears. Thus the differential equations

$$\frac{dy}{dx} = y - x^2 \tag{1-2}$$

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0 \tag{1-3}$$

and

$$\frac{d^2s}{dt^2} - 3\frac{ds}{dt} + 2s = t^2 \tag{1-4}$$

are of the first, third, and second orders, respectively.

The study of differential equations is important because of the frequency with which they arise in the applications of mathematics to scientific problems. The student has already found in his study of the calculus that the derivative appears in a great variety of problems—as the slope of a curve, as a velocity or acceleration in the study of motion, as the rate of change of some function in a great many connections. Now, in the exact sciences a vast number of problems arise in which the quantity whose value is sought is known only through some relation satisfied by its derivative. Thus the velocity or acceleration of a moving body may be known and the distance traveled in a given time required; or the rate at which a quantity is increasing or decreasing may be given and the magnitude of the quantity itself be sought. In such cases the conditions of the problem supply us with a differential equation satisfied by the unknown function, and we are faced with the problem of finding what the function is. The process of finding the function that satisfies a differential equation is called *solving* the equation.

The differential equations at the beginning of this section are of a particular kind. Each contains one independent variable and one dependent variable (or function). More generally, *a differential equation is an equation connecting certain independent variables, certain functions (dependent variables) of these variables, and certain derivatives of these functions with respect to the independent variables.* Differential equations are divided into classes according as there are one or more independent variables.

If there is a single independent variable, so that the derivatives are ordinary derivatives, the equation is called an *ordinary differential equation.*

If there are two or more independent variables, so that the derivatives are partial derivatives, the equation is called a *partial differential equation.* Thus

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = yz$$

is a partial differential equation. Here z, the dependent variable, is a function of the two independent variables x and y.

In later chapters we shall consider simultaneous equations in which there are two or more dependent variables satisfying two or more differential equations. For example, it might be proposed to find two functions, $x(t)$ and $y(t)$, satisfying the equations

$$\frac{dx}{dt} = 2y + x \qquad \frac{dy}{dt} = 3y + 4x$$

In the early part of our study, however, we shall be concerned with a single equation with one dependent variable. We shall begin with the simplest case, the equation of the first order. This equation, when solved for the derivative, appears in the form

$$\frac{dy}{dx} = f(x,y)$$

1-2. Solutions of Differential Equations. A relation $y = g(x)$ is a *solution* or *integral* of (1-1) if

$$f[x,g(x),g'(x), \ldots ,g^{(n)}(x)] \equiv 0$$

that is, y is such a function of x that if y and its derivatives be expressed in terms of x and substituted into the differential equation, the equation is identically satisfied. Thus $y = x^2 + 2x + 2$ is a solution of (1-2), for on making the substitution we have $2x + 2 = x^2 + 2x + 2 - x^2$, which holds for all values of x. Similarly, $y = e^x$ is an integral of (1-3), for on substituting in (1-3) we have the identity $e^x - e^x = 0$.

It is frequently neither convenient nor desirable to express the dependent variable in terms of the independent variable. An implicit relation,

$F(x,y) = 0$, is a solution, if when solved explicitly for y in terms of x, it yields a solution in the way described above. However, the implicit relation can be differentiated and the derivatives found in terms of x and y and tested by substitution in the differential equation without the necessity of solving explicitly. A test can thus be made when it is difficult or altogether impossible to solve for y in terms of x. For example, let us show that

$$x^2 = 2y^2 \log y$$

is a solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

Differentiating the proposed solution, we have

$$2x = (2y + 4y \log y) \frac{dy}{dx}$$

Solving for dy/dx and replacing $\log y$ by its value, $x^2/2y^2$,

$$\frac{dy}{dx} = \frac{x}{y + 2y \log y} = \frac{x}{y + \dfrac{x^2}{y}} = \frac{xy}{x^2 + y^2}$$

The differential equation is satisfied.

By differentiating and substituting in the differential equation we can test whether a given relation is a solution of a given differential equation, but we have as yet no clue as to how the solution is found. A considerable part of our further study will consist in devising methods of finding solutions of particular classes of equations.

The student has already had practice in solving differential equations of a particularly simple form. The problem of integration is to find a function whose derivative is a given function of the independent variable

$$\frac{dy}{dx} = f(x)$$

The most general solution of this differential equation is

$$y = \int f(x)\, dx + C$$

where C is an arbitrary constant.

We shall now consider a simple problem in which a method of solution readily occurs to us.

Problem. The function e^x has the property that the derivative is equal to the function. Find the most general function with this property.

Let y be the function; then the required property is expressed by the differential equation

$$\frac{dy}{dx} = y \tag{1-5}$$

One solution of this is obviously $y = 0$. If $y \neq 0$, we can divide by y and put the equation in the form

$$\frac{dy}{y} = dx$$

The first member is the differential of $\log y$ or $\log(-y)$, according as y is positive or negative; and the second is the differential of x. Since two functions whose differentials are equal differ at most by a constant, we have, integrating,

$$\log(\pm y) = x + C$$

or

$$y = \pm e^{x+C}$$

where C is an arbitrary constant. This value of y, together with $y = 0$, gives all functions with the required property.

We can put the result in a different form by setting

$$\pm e^{C} = K$$

thus changing the form of the constant. Then

$$y = Ke^{x} \tag{1-6}$$

1-3. First Method of Solution. Variables Separable. The solution of the preceding problem was effected by writing the equation in two terms, one of which is a function of x alone, the other a function of y alone, from which an integration gave the solution at once. If a differential equation can be written in the form

$$M(x)\,dx + N(y)\,dy = 0 \tag{1-7}$$

where, as the notation indicates, M is a function of x alone and N a function of y alone, the solution is

$$\int M(x)\,dx + \int N(y)\,dy = C \tag{1-8}$$

where C is an arbitrary constant. The problem is then reduced to the problem of evaluating the two integrals in (1-8). In Eq. (1-7) we say that the variables are *separated.*

It is clear that we can separate the variables in only a limited class of differential equations. It happens, however, that in many of the simpler equations met with in the applications of mathematics the variables can be separated, so that the method is one of importance.

Example. Solve the equation

$$\frac{dy}{dx} = \frac{x}{y\sqrt{1 - x^2}}$$

Separating the variables,

$$y\,dy - \frac{x\,dx}{\sqrt{1 - x^2}} = 0$$

Integrating,

$$\tfrac{1}{2}y^2 + \sqrt{1 - x^2} = C$$

or

$$y^2 + 2\sqrt{1 - x^2} = C' \qquad C' = 2C$$

1-4. Arbitrary Constants. The number of solutions of Eq. (1-7) is infinite, since C in (1-8) may be given any value. The infinitude of its solutions is characteristic of a differential equation. In the process of solution of a differential equation of the first order there comes a step by which the differentials or derivatives are removed by an integration, and this integration introduces an arbitrary constant.[1] The solution containing this arbitrary constant is called the *general solution* of the equation. A solution which results from giving a particular value to the arbitrary constant is called a *particular solution*. Thus $y = 3e^x$ and $y = -2e^x$ are particular solutions of (1-5).

By saying that a constant is arbitrary we mean that it can be given any value within a certain range of values. Frequently any value whatever may be given the constant; sometimes only a limited range of values will yield real solutions. For example, in $y = Cx$, C may have any value; in $x^2 + y^2 = C$ only positive values of C give y as a real function of x. In simplifying the solution of a differential equation it is advantageous to replace a function of an arbitrary constant by a new constant since the function is itself an arbitrary constant. Thus, in the solution of (1-5) we replaced $\pm e^C$ by the simpler constant K.

In general, a function of one or more arbitrary constants is itself an arbitrary constant. An expression may have more apparent arbitrary constants than essential ones, for we may be able to replace the constants that appear by a smaller number. For example, $y = Ke^{x+C}$ is a solution of (1-5) with two constants, but if we replace Ke^C by the new constant C', we have $y = C'e^x$ in which there is a single constant. Again

$$y = x^2 + A + B$$

is no more general than $y = x^2 + C$, for to give arbitrary values to A and B is equivalent to giving arbitrary values to C. A less obvious case is the equation

$$x^3y^3 + C_1x^2y^2 + C_2xy + C_3 = 0$$

which contains three constants. But this is a cubic equation in xy whose solution is some function of the coefficients

$$xy = f(C_1,C_2,C_3)$$

and this can be written in the equally general form

$$xy = C$$

[1] The presence of an arbitrary constant in the solution will be given a rigorous demonstration later.

It will be found that an equation of the nth order has a solution containing n essential arbitrary constants. Such a solution will be called a *general solution*. A solution obtained by giving particular values to the constants is a *particular solution*.

1-5. Solutions Satisfying Specified Conditions. Owing to the presence of the arbitrary constant in the general solution of the differential equation of the first order, we are able to make the solution satisfy one condition by particularizing the constant. The commonest form of the condition is that the dependent variable shall have a specified value for a given value of the independent variable. This condition is satisfied by substituting the given values in the general solution and solving for the constant. Thus the solution of (1-5) such that $y = 1$ when $x = 0$ is found by setting $x = 0$, $y = 1$ in the general solution (1-6),

$$1 = K$$

whence

$$y = e^x$$

is the solution with the required property.

A condition to be satisfied by the solution may appear in various other ways. For example, find a solution of (1-5) such that y has a value 1 greater at $x = 1$ than it has at $x = 0$. This gives, substituting in (1-6),

$$Ke = K + 1 \qquad \text{or} \qquad K = \frac{1}{e - 1}$$

The solution is

$$y = \frac{e^x}{e - 1}$$

The method of procedure in any case is to express the required condition as an equation in which the arbitrary constant appears. From this equation the value of the constant is determined. Sometimes, of course, no value of the constant will satisfy the equation, in which case there is no solution with the required property. Sometimes, also, several values of the constant are determined, and there are several solutions with the required property.

1-6. Velocities. Velocities, and rates of change generally, are derivatives. Let an object be moving along a path. At time t let its distance from a fixed point O of the path, measured along the path, be s (see Fig. 8). By convention s will be positive on one side of O and negative on the other. At time $t + \Delta t$ let its distance be $s + \Delta s$. The average velocity for this period of time Δt is $\Delta s/\Delta t$. The instantaneous velocity v at time t is the limit of this ratio,

$$v = \frac{ds}{dt}$$

If v is positive, the object is moving as t increases in the direction of the positive end of the path; if negative, the motion is in the opposite direction.

Suppose that the velocity is given in terms of s, or t, or both. We have then a differential equation whose solution—a relation between s and t—will enable us to locate the object at a given time. An example will make the matter clear.

Problem. I live on a straight road 6 miles due north of school and I leave home going south at a speed of 30 miles an hour. If my velocity is proportional to the square of my distance from school, find my motion. When, if ever, will I reach school?

Let s be my distance from school t hr after I leave home, distances north from school being considered positive. We are given that

$$v = \frac{ds}{dt} = ks^2$$

The factor of proportionality k is determined by the knowledge that $v = -30$ when $s = 6$,

$$-30 = 36k \qquad k = -\frac{5}{6}$$

The differential equation of the motion is then

$$\frac{ds}{dt} = -\frac{5}{6}s^2$$

We solve this by separating variables,

$$-\frac{ds}{s^2} = \frac{5}{6}\,dt$$

whence, integrating,

$$\frac{1}{s} = \frac{5}{6}t + C$$

When $t = 0$, $s = 6$, whence $C = \frac{1}{6}$. We have, finally

$$s = \frac{6}{5t + 1}$$

This gives my precise position at any time.

Since no value of t will give $s = 0$, the school is never reached.

EXERCISES

Solve the problem of the journey to school under the following six hypotheses about the velocity, the other conditions remaining the same.

1. The velocity is proportional to the distance from school.

2. The velocity is proportional to the square root of the distance from school.

3. The velocity is inversely proportional to the distance from school.

4. The velocity is proportional to the distance from a tower 4 miles south of school.

5. The velocity is proportional to the square of the distance from the tower.