MODERN TRIGONOMETRY

CAVANAGH

Modern Trigonometry

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Table I, "Values of Circular Functions," Table II, "Values of Trigonometric Functions," and Table III, "Common Logarithms," in the Appendix are from *Modern College Algebra and Trigonometry* by Edwin F. Beckenbach and Irving Drooyan. © 1964, 1965, 1966, 1968 by Wadsworth Publishing Company, Inc., Belmont, California. Reprinted by permission of the authors and publisher. Table IV, "Logarithms of Values of Trigonometric Functions," is from *Modern Trigonometry,* 2nd. ed., by William Wooton, Edwin F. Beckenbach, and Mary P. Dolciani. Copyright © 1966, 1969 by Houghton Mifflin Company, Boston. Reprinted by permission of the publisher.

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Modern Trigonometry

Preface

Trigonometry, once an almost exclusive science for navigators and surveyors, has expanded into virtually all fields of science and engineering. As its uses have become less specific, trigonometry has increasingly become an analytic subject. Therefore, Modern Trigonometry, as its name implies, gives primary consideration to the analytic aspects of the field, although the computational aspects are not overlooked.

This book is a concise coverage of the subject designed for students who have studied college algebra or its equivalent. It will fit nicely into a one-semester course meeting three times a week, or a quarter course that meets daily. To cover the book in a shorter time would necessitate the omission of some topics. A teacher's manual that includes solutions for all exercises also gives suggestions for pacing courses of different lengths.

The function concept is emphasized through the text. Periodic functions are discussed in Chapter 0, and trigonometric functions are introduced in Chapter 1. The concept of trigonometric functions of real numbers is discussed in terms of "wrapping" the real number line around the unit circle. Inverse trigonometric functions are the subject of Chapter 4. Chapter 5, on the solutions of triangles, also relates the definitions of the trigonometric functions of real numbers to the ratios of different sides of a right triangle.

Chapter 6 explores complex numbers through De Moivre's theorem. The rectangular and polar forms for the representation of a complex number are compared.

Color is used throughout to stress basic concepts introduced in the text and to focus attention on parts of graphs being emphasized.

Chapter 0, as its number implies, will be review material for many students and, as such, may be covered very quickly in some classes. However, all classes should cover the last two sections, which discuss the idea of periodic functions.

The answer section for Chapter 2 contains proofs of some of the identities in the exercises. There are many carefully worked identities included as examples in the chapter. The emphasis in Chapter 3 is on the general solution of trigono-

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metric equations, but in a way that principal solutions could be emphasized readily by the teacher who prefers to do so.

An abundance of exercises is provided in each chapter—more than could be assigned to one class—to give teachers the opportunity to vary assignments from class to class. Each chapter ends with a set of exercises for review.

I am indebted to Professors Rex Schweers and Forest Fisch of Colorado State College for their help in class testing the material in this book and the helpful suggestions that resulted from the testing. Thanks also go to James L. Jackson, College of San Mateo, and Raymond C. Strauss, De Anza College, who reviewed the manuscript, and Joseph Teeters, a graduate student at Colorado State College who helped prepare the selected answers. Finally, I thank Berniece Zimmerman and Susan Phelps for the many long hours they spent in typing the manuscript.

Contents

0

Sets, relations, and functions

0.1 Introduction

The science of trigonometry was originally concerned with computing the unknown sides and angles of triangles. As such, trigonometry was an important subject in colleges and universities during the early days of our country, for many students later applied their mathematics to surveying and navigation. So, even then, trigonometry, unlike some mathematical studies, had very practical applications. As science and mathematics have developed, trigonometric concepts have been broadened, and trigonometry now is usually treated from an analytic viewpoint. Trigonometric analysis today is an important tool in most branches of science and engineering. Therefore, this book gives primary consideration to the analytic aspects of the subject, but the computational aspects are also considered.

0.2 Sets

One of the unifying ideas in mathematics is the idea of **set.** Basically, we think of a set as being **a collection of objects.** We shall assume that there is some way to tell whether a particular object is in the set under consideration. The objects in sets are called **elements.**

Here, lowercase letters, such as a, b, and x, and other symbols, will represent the elements in sets. Uppercase letters, such as A, C, and Q, will represent sets. The symbol $\in$, derived from the Greek letter ϵ, is used to indicate that a particular element is in a set. Thus $x \in A$ means that the element x is in the set A; and $x \in A$ may be read as *x is in A* or *x belongs to A*. If the set A consists of several elements, such as a, $*$, and 6, we can indicate it by $a \in A$, $* \in A$, $6 \in A$. Since this could become quite cumbersome, we use braces to indicate the same idea and write $A = \{a, *, 6\}$, which states that A consists of a, $*$, and 6.

It is important to consider the relationships between sets. One of these is that of one set being a **subset** of another set.

> *A set A is a **subset** of a set B if and only if every element in the set A is also in the set B. This is designated by $A \subseteq B$, and is read "A is a subset of B."*

There is a special kind of subset called a **proper subset.**

> *A is a **proper subset** of B if and only if $A \subseteq B$ and there exists at least one element in B which is not in A. This is designated by $A \subset B$.*

Thus, every subset of B except B itself is a proper subset of B.

A diagonal slash, /, drawn through certain symbols expressing relationships, is used to indicate that these relationships do not hold. Thus, $A \nsubseteq B$ is read *A is not a subset of B*, and $x \notin A$ is read *x does not belong to A*.

$$\{\bigstar, 0\} \subset \{\bigstar, 0, \square\} \qquad \{\bigstar, 0\} = \{0, \bigstar\}$$

$$\{\bigstar, 0, \square\} \nsubseteq \{\bigstar, 0\} \qquad \{\bigstar, 0, \square\} \subseteq \{0, \square, \bigstar\}$$

$$0 \in \{\bigstar, 0\} \qquad \square \notin \{\bigstar, 0\} \qquad 0 \neq \{0\}$$

Figure 0.1

The idea of **equality**, important in all of mathematics, also expresses an important relationship between sets. *Two sets, A and B, could be said to be equal if and only if they consist of exactly the same elements.* This would be a satisfactory definition of equality of sets, and it is a definition which is often used. However, we shall use a different definition of equality of sets, although it will carry the same meaning.

> *Two sets A and B are **equal** if and only if $A \subseteq B$ and $B \subseteq A$. Equality is designated by $A = B$.*

We will observe that $\{*, \circ\} = \{\circ, *\}$. We also use $A \neq B$ to indicate that *A is not equal to B*. Thus, $\{*, \circ\} \neq \{*, \square, \circ\}$, since $\{*, \square, \circ\} \nsubseteq \{*, \circ\}$. Examples of the symbols used to express relationships between sets are shown in Figure 0.1.

A very special set in mathematics is called the **empty set.**

*The **empty set** is the set which does not contain any elements. It is designated by ∅, which is read "the empty set" or "the null set."*

The empty set is considered to be a subset of every set and a proper subset of every set except itself.

Sometimes we refer to a relationship between two sets by saying that the two sets are in **one-to-one correspondence.** Intuitively, this means that the two sets have exactly the same number of elements. This idea is only appropriate for finite sets, so we define one-to-one correspondence in a different way.

*Two sets are in **one-to-one correspondence** if and only if the elements of the two sets can be matched so that each element of one set corresponds to exactly one element in the other set.*

We can demonstrate that two sets are in one-to-one correspondence by showing how the elements of the two sets are matched. This is illustrated in Figure 0.2 with the sets $A = \{*, \circ, \triangle, \square\}$ and $B = \{a, e, i, u\}$. Two different ways of matching the elements in sets A and B are shown in the Figure. One such matching would be sufficient to demonstrate a one-to-one correspondence between the two sets. You can probably illustrate several other similar matchings between the two sets.

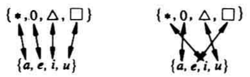

Figure 0.2

Another useful way of discussing sets is illustrated by $\{x \mid x$ **is a whole number between 2 and 6**$\}$, which is read *the set of all x such that x is a whole number between* 2 *and* 6. This collection of symbols, called **set-builder notation**, is used extensively in this book. It names a variable (x in this case) and states a condition which the variable must satisfy (in this case, x is a whole number between 2 and 6) in order to belong to the particular set under consideration. We note that

$$\{x \mid x \text{ is a whole number between 2 and 6}\} = \{3, 4, 5\}.$$

Another example of the set-builder notation is

$$\{1, 3, 5, 7\} = \{y \mid y \text{ is one of the first four odd natural numbers}\}.$$

The set-builder notation is particularly useful when one is dealing with large finite sets or with infinite sets.

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Exercises 0.1

1. Let $A = \{1, 2\}$.
   a. List all the subsets of A that contain 0 elements.
   b. List all the subsets of A that contain 1 element.
   c. List all the subsets of A that contain 2 elements.
   d. How many subsets does A have?
   e. List all the proper subsets of A.

2. Let $B = \{\#, \$, \circ\}$.
   a. List all the subsets of B that contain 0 elements.
   b. List all the subsets of B that contain 1 element.
   c. List all the subsets of B that contain 2 elements.
   d. List all the subsets of B that contain 3 elements.
   e. How many subsets does B have?
   f. List all the proper subsets of B.

3. Let $C = \{4, 5, 6, 7\}$.
   a. List all the subsets of C that contain 0 elements.
   b. List all the subsets of C that contain 1 element.
   c. List all the subsets of C that contain 2 elements.
   d. List all the subsets of C that contain 3 elements.
   e. List all the subsets of C that contain 4 elements.
   f. How many subsets does C have?
   g. List all the proper subsets of C.

4. Some of the information in the table below comes from Exercises 1–3. Fill in the rest of the table, if you can.

number of elements in set	0	1	2	3	4	5	6	n
number of subsets			4	8	16			

5. Consider the following sets: $A = \{1, 2, 3\}$, $B = \{1, 3\}$, $C = \{1, 2, 4\}$, $\emptyset$.
   a. Indicate, with the appropriate symbols, all the subset relationships among the four sets.
   b. Indicate, with the appropriate symbols, all the proper subset relationships among the four sets.
   c. Indicate, with the appropriate symbols, all cases in which one set is *not* a subset of another set.

In Exercises 6–9, indicate, with the appropriate symbols, whether or not the two given sets are equal.

6. $A = \{5\}$, $B = \{9\}$.
7. $A = \{5, 7, 11\}$, $B = \{11, 7, 5\}$.
8. $A = \{4, 5, 6, 7\}$, $B = \{x \mid x$ is a natural number between 3 and 8$\}$.
9. $A = \{0\}$, $\emptyset$.

Indicate whether a one-to-one correspondence exists between the two given sets.

10. $A = \{5\}$, $B = \{9\}$.
11. $A = \{2, 3, 4\}$, $B = \{2, 4\}$.
12. $A = \{1, 3, 15\}$, $B = \{x \mid x$ is an odd natural number less than 6$\}$.
13. $A = \{1, 2, 3\}$, $B = \emptyset$.

In Problems 14–17, replace the comma between set symbols with either $\in$ or $\notin$ so as to form a true statement.

14. $\emptyset$, $\{1, 2, 3\}$
15. $\{7\}$, $\{4, 5, 6, 7\}$
16. 6, $\{6, 7, 8\}$
17. 5, $\{x \mid x$ is an odd natural number$\}$

Designate each of the sets by using braces and listing the members.

Example

$\{x \mid x$ is a natural number between 3 and 7$\}$

Solution

$\{4, 5, 6\}$

18. $\{x \mid x$ is a day in the week$\}$
19. $\{x \mid x$ is a natural number between 15 and 21$\}$
20. $\{z \mid z$ is a digit in your home address$\}$
21. $\{y \mid y$ is a month in the year$\}$
22. $\{t \mid t$ is a digit in your age in years$\}$
23. $\{w \mid w$ is a natural number less than 7$\}$

Designate each set by using the set-builder notation.

Example

$\{1, 2, 3\}$

Solution

$\{x \mid x$ is one of the first three natural numbers$\}$

24. $\{2, 4, 6, 8\}$
25. $\{5, 6, 7, 8, 9, 10\}$

26. {2, 4, 8, 16}
27. {April, June, September, November}

0.3 Operations on sets

There are several operations on sets which are used frequently. One of these is the **union** of two sets.

The ***union*** *of sets A and B is the set of all the elements which are in set A, in set B, or in both sets A and B. This is designated by* $A \cup B$ *and is read "the union of A and B" or sometimes "A cup B."*

Thus, if $A = \{1, 3\}$, $B = \{3, *, \circ\}$, and $C = \{g, h\}$, then $A \cup B = \{1, 3, *, \circ\}$ and $A \cup C = \{1, 3, g, h\}$. If A is any set, $A \cup A = A$ and $A \cup \emptyset = A$. Why?

A second important set operation is the **intersection** of two sets.

The ***intersection*** *of sets A and B is the set of all the elements which are in both set A and set B. This is designated by* $A \cap B$ *and is read "the intersection of A and B" or, sometimes, "A cap B."*

Thus, if $A = \{1, 3\}$, $B = \{3, *, \circ\}$, and $C = \{g, h\}$, then $A \cap B = \{3\}$ and $A \cap C = \emptyset$. If A is any set, $A \cap \emptyset = \emptyset$ and $A \cap A = A$. Why?

Two nonempty sets A and B are said to be ***disjoint*** *if and only if* $A \cap B = \emptyset$.

Thus, in the above example, A and C are disjoint sets and B and C are disjoint sets, but A and B are not disjoint sets.

In many cases we consider only subsets of some particular set. This particular set under consideration is called **the universal set** or, sometimes, **the universe.** The universal set is usually designated by U. Let A be any subset of a universal set U. There is another subset of U which is closely related to A. This set is called the **complement of A.**

If A is a subset of U, the set of all the elements in U which are not in A is called ***the complement of A****. The complement of A is designated by* A'.

For example, if $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 5\}$, and $B = \{3\}$, then $A' = \{2, 3, 4\}$ and $B' = \{1, 2, 4, 5\}$.

Closely related to the complement of a set A is the idea of the **difference** of two sets.

For any two sets A and B, the ***difference*** *of A and B is the set of all the elements which are in A and not in B. This is designated by* $A - B$.

If $C = \{a, e, o\}$, $D = \{o, u\}$ and $E = \{a, e\}$, then $C - D = \{a, e\}$, $D - C = \{u\}$, and $D - E = \{o, u\}$. Note that it is not necessary for B to be a subset of A to discuss the difference of A and B. If A and B are subsets of a universal set U, then $A - B = A \cap B'$. *Why?*

Exercises 0.2

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 4, 8\}$, and $D = \{1, 2, 3\}$. Perform the following set operations.

1. A'
2. B'
3. C'
4. D'
5. $A \cup B$
6. $A \cap B$
7. $A - B$
8. $(A \cup B)'$
9. $(A \cap B)'$
10. $A' \cup B'$
11. $A' \cap B'$
12. $A \cap B'$
13. $A \cap C$
14. $A - C$
15. $C - A$
16. $C \cap B$
17. $C - B$
18. $A - D$
19. $C - U$
20. $D - U$
21. $B - U$
22. $A \cup B'$
23. $C' \cap D'$
24. $D - C$
25. $A' - C'$
26. $C \cap A'$
27. $(D')'$
28. $(B')'$
29. $(A \cap B) \cup C$
30. $(D' - B) \cap C$
31. $(A \cup C) \cap (B \cup C)$
32. $A - (B \cup C)'$
33. $(B \cup C) \cap D$
34. $(B \cap D) \cup (C \cap D)$
35. $(A - B)' \cup D'$
36. $(A \cup B) \cap C$

For problems 37–42, let A, B, and C be any sets.

37. Is $A \cup B = B \cup A$? Why?
38. Is $A \cap B = B \cap A$? Why?
39. Is $A - B = B - A$? Why?
40. Can $A - B = B - A$? Why?
41. Is $(A \cup B) \cup C = A \cup (B \cup C)$? Why?
42. Is $(A \cap B) \cap C = A \cap (B \cap C)$? Why?

0.4 Our number system

You are familiar with real numbers from studying arithmetic and algebra. Some real numbers are called **natural numbers.** Other subsets of the real numbers are called **integers, rational numbers,** and **irrational numbers.** In this section we shall briefly consider some of the different sets of numbers involved in the development of the real-number system.

1. The **set of natural numbers,** denoted by N, whose elements are the familiar "counting" numbers.

$$N = \{1, 2, 3, \ldots\}.$$

2. The **set of whole numbers,** denoted by W, which consists of the set of natural numbers and 0.

$$W = \{0\} \cup N = \{0, 1, 2, 3, \ldots\}.$$

3. The **set of integers,** denoted by J, whose elements are the natural numbers, their additive inverses, and 0.

$$J = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

4. The **set of rational numbers,** denoted by Q, which consists of all those numbers that can be represented as the quotient of two integers where the divisor is not 0. Among the elements of Q are such numbers as $-\frac{2}{3}, \frac{18}{24}, \frac{7}{3}, -\frac{5}{1}$, and $\frac{0}{5}$. Thus,

$$Q = \left\{x \mid x = \frac{a}{b}, a \in J, b \in J, b \neq 0\right\}.$$

It can be proved, although we shall not do so here, that all rational numbers can be expressed as **terminating decimals** or as **repeating decimals.** Examples of these are $1/4 = .25$, a terminating decimal, and $1/11 = .0909\overline{09}$, a repeating decimal. Conversely, every number that is a terminating decimal or a repeating decimal is a rational number.

5. The **set of irrational numbers,** denoted by H, which consists of all numbers with decimal expansions which are nonrepeating and nonterminating. Among the elements of this set are such numbers as $\sqrt{2}$, $-\sqrt{3}$, and π.

$$H = \{x \mid x \text{ is an irrational number}\}.$$

6. The **set of real numbers,** denoted by R, which is the union of the set of rational numbers and the set of irrational numbers.

$$R = Q \cup H.$$

7. The **set of complex numbers,** denoted by C, which consists of all numbers that can be represented in the form $x + yi$, where x and y are real numbers and $i = \sqrt{-1}$. Thus,

$$C = \{z \mid z = x + yi, x \in R, y \in R, i = \sqrt{-1}\}.$$

Complex numbers with $x = 0$, which may be represented in the form yi, are called **pure imaginary numbers.**

In a course in modern algebra or the structure of numbers, distinctions would be made among the sets of numbers that have not been made here. This is intended only as a quick review of material with which you should be familiar, not as a thorough development of our number system.

A **real-number line** can be drawn as follows. Draw any line and pick two points on the line. Call the point on the left 0 and the point on the right 1. The natural numbers can be assigned to the points *at unit intervals past one*, and their additive inverses can be assigned to points *at unit intervals to the left of* 0. It is easy to determine the point half-way between 0 and 1 and call that point 1/2. In a similar fashion the other rational and irrational numbers can be assigned to points on the line. It can be proved that there is a one-to-one correspondence between the set of points on the line and the set of real numbers, but we shall not do so here.

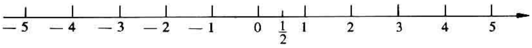

Figure 0.3

Exercises 0.3

Express the following fractions as decimals.

1. $\frac{1}{5}$ 2. $\frac{3}{4}$ 3. $\frac{2}{3}$ 4. $\frac{5}{6}$ 5. $\frac{3}{11}$

6. $\frac{4}{15}$ 7. $\frac{5}{7}$ 8. $\frac{7}{8}$ 9. $\frac{29}{11}$ 10. $\frac{13}{6}$

In Exercises 11–22, express the decimals as fractions. Reduce to lowest terms.

Examples

a. $.375$ b. $.23\overline{23}$ c. $2.354\overline{354}$

Solutions

a.
$$.375 = \frac{375}{1000} = \frac{3(125)}{8(125)} = \frac{3}{8}.$$

b.
$$\text{Let } x = .23\overline{23}.$$
$$100x = 23.23\overline{23},$$
$$100x - x = 23.00\overline{00},$$
$$99x = 23,$$
$$x = \frac{23}{99}.$$

c.
$$\text{Let } x = 2.354\overline{354}.$$
$$\text{then } 1000x = 2354.354\overline{354},$$
$$1000x - x = 2352,$$
$$x = \frac{2352}{999}, = \frac{3(784)}{3(333)} = \frac{784}{333}.$$

11. .235 12. .3125 13. 16.1

14. 5.472 15. $.45\overline{45}$ 16. $.29\overline{29}$

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