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Ravinder R. Puri

Mathematical Methods of Quantum Optics

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With 13 Figures

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Dedicated to
My Inspiration – My Wife
Shyama

Preface

This book is intended to provide a much needed systematic exposition of the mathematical methods of quantum optics, something that is not found in existing books. It is primarily addressed to researchers who are new to the field. The emphasis, therefore, is on a simple and self-contained, yet concise, presentation. It provides a unified view of the concepts and the methods of quantum optics and aims to prepare a reader to handle specific situations. A number of formulae scattered throughout the scientific literature are also brought together in a natural manner.

The broad plan of the book is to introduce first the basic physics and mathematical concepts, then to apply them to construct the model hamiltonians of the atom–field interaction and the master equation for an atom–field system interacting with the environment, and to analyze the equations so obtained. A brief description of the contents of the chapters is as follows.

The first chapter introduces the basic postulates of quantum mechanics, brings out their implications and develops the associated operational techniques. It discusses the measurement problem, the paradoxes of quantum mechanics and the local hidden variables theory, since quantum optics provides experimental means of examining these issues. Chapter 2 outlines the algebra of the exponential operator, which plays a prominent role in mathematical physics. The concept of Lie algebra is introduced and the standard hamiltonians of quantum optics are treated as elements of one or the other finite-dimensional Lie algebra. The question of representations of Lie algebras is addressed in Chap. 3. The notion of coherent states emerges as a continuous representation of a Lie algebra. The concept of quasiprobabilities is developed in Chap. 4. Their usefulness as operational tools and as entities for identifying purely quantum effects is demonstrated. Chapter 5 presents the essential elements of the theory of stochastic processes. The theory of classical and quantized electromagnetic (e.m.) fields is outlined in Chap. 6. It describes the characterization of the e.m. field in terms of its correlation functions and also their role in identifying the signatures of field quantization.

By starting with the hamiltonian for an atom interacting with the e.m. field in the dipole approximation, Chap. 7 describes ways of reducing it to simpler, mathematically tractable forms commensurate with given physical conditions. The standard models of quantum optics are thereby derived. The effects of the environment on an atom–field system are the subject of the quantum theory of damping outlined in Chap. 8. Here the master equation for the evolution of a system in contact with a reservoir is constructed and methods of solving it are discussed.

Chapter 9 analyzes the perturbative solution of the master equation of an atomic system in an external field. This leads to the notions of susceptibility, multiwave mixing and the absorption spectrum.

The method of solving a set of linear equations with time-independent coefficients in terms of generalized eigenvectors is outlined in Chap. 10. That chapter presents the solution of a two-term recurrence relation and identifies and solves exactly solvable quadratic three-term recurrence relations. These recurrence relations encompass many well-known quantum optical situations.

Chapters 11–14 deal with the solution of some standard model systems. Chapter 11 identifies the class of analytically exactly solvable models of an effective two-level atom and that of an effective three-level atom in a quantized field. It provides a unified treatment of the exactly solvable hamiltonians of quantum optics.

The problem of an externally driven two-level atomic system dissipating into a squeezed reservoir is addressed in Chap. 12. The exactly solvable cases of an arbitrary time-dependent drive are identified. The exact dynamics in a monochromatic drive is investigated and the collective effects in a driven two-level atomic system are highlighted. Chapter 12 also briefly discusses the dynamical behaviour of a three-level atom dissipating into a reservoir at absolute zero temperature and reveals the effects of almost equally spaced pairs of energy levels.

The dynamics of a field dissipating into a linear or two-photon non-linear reservoir is the subject of Chap. 13. The evolution of an atomic system interacting with a single damped quantized cavity mode is investigated in Chap. 14. This chapter also outlines the theory of the micromaser.

I am indebted to Girish Agarwal for teaching me the subject of quantum optics. Valuable contributions to my understanding have been gained through my association with Robert (Robin) Bullough, Joseph Eberly, Fritz Haake, Shoukry Hassan, Rajiah Simon, Subhash Chaturvedi, V. Srinivasan, Subhasish Dattagupta, Surya Tiwari, Dinkar Khandekar and Suresh Lawande. I am grateful to Debabrata Biswas and Aditi Ray for their valuable suggestions and help in preparing the manuscript. I am thankful to Dinesh Sahni for his support and encouragement. Angela Lahee of Springer-Verlag deserves a big thank you for her careful editing.

Mumbai,
January 2001

Ravinder Puri

Contents

1. Basic Quantum Mechanics

Quantum optics is the quantum theory of interaction of the electromagnetic field with matter. In this chapter we recapitulate basic concepts and operational methods of the quantum theory essential for developing the theory of quantum optics. We delve also in to the controversial issue of interpretation of the quantum theory as a classical statistical theory. Quantum optics provides means for subjecting these conceptually controversial issues to experimental tests.

1.1 Postulates of Quantum Mechanics

In this section we state five basic postulates of Quantum Mechanics and discuss some of their important implications.

1.1.1 Postulate 1

An isolated quantum system is described by a vector in a Hilbert space. Two vectors differing only by a multiplying constant represent the same physical state.

Following the notation introduced by Dirac [1], we represent a vector by a *ket*, $|\ \rangle$.

A Hilbert space is a complex linear vector space equipped with the definition of a scalar product and spanned by a complete set of vectors [2]. The meaning and implications of these properties of the Hilbert space are explained below. They are crucial for relating the theory with experimental observations.

Linear Vector Space. A Hilbert space is a complex linear vector space. We assume familiarity with the notion of a linear vector space over the field of complex numbers (c-numbers) [2]. We recall that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are vectors in a complex linear vector space then a *linear combination* $\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$ for arbitrary complex numbers α_1, α_2 is also a vector in the same space. A set of vectors $|\psi_1\rangle, \cdots, |\psi_n\rangle$ is said to be *linearly independent* if

$$\sum_{i=i}^{n} \alpha_i|\psi_i\rangle = 0, \tag{1.1}$$

implies $\alpha_i = 0$ for all $i = 1, \ldots, n$. The maximum number of linearly independent vectors in a linear vector space is called its *dimension*.

Scalar Product. To say that the Hilbert space is a *Euclidean or scalar product space* means that it is possible to associate with every pair of vectors $|\phi\rangle$ and $|\psi\rangle$ in it a complex number, denoted by $\langle\phi|\psi\rangle$, such that

1. $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$, where $*$ denotes the operation of complex conjugation;
2. If $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$ then $\langle\phi|\psi\rangle = c_1\langle\phi|\psi_1\rangle + c_2\langle\phi|\psi_2\rangle$;
3. $\langle\psi|\psi\rangle > 0$;
4. $\langle\psi|\psi\rangle = 0$ if and only if (iff) $|\psi\rangle = 0$.

In the following we list some consequences of these axioms.

- The scalar product associates with a vector $|\ \rangle$ its *dual* $\langle\ |$ called a *bra* [1].
- The non-zero positive number $|||\psi\rangle|| \equiv \sqrt{\langle\psi|\psi\rangle}$ is called the *norm* or the *length* of the vector. Since two vectors differing only by a multiplication factor represent the same physical state, we can represent a physical state by a vector of a fixed, say unit, norm if the norm is finite. Hence, $|\psi\rangle$ is physically an acceptable vector if its norm is finite i.e. if

$$\langle\psi|\psi\rangle < \infty. \tag{1.2}$$

- The vector $|\phi\rangle\langle\phi|\psi\rangle$ is the *projection* of a vector $|\psi\rangle$ along the vector $|\phi\rangle$. The scalar product $\langle\phi|\psi\rangle$ is a measure of the overlap between the vectors $|\psi\rangle$ and $|\phi\rangle$. If $\langle\phi|\psi\rangle = 0$ then $|\psi\rangle$ and $|\phi\rangle$ are said to be *orthogonal* to each other.
- Two sets of vectors $|\psi_1\rangle, \cdots, |\psi_n\rangle$ and $|\phi_1\rangle, \cdots, |\phi_n\rangle$ are said to be orthonormal to each other if

$$\langle\phi_i|\psi_j\rangle = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.3}$$

- A set $|e_1\rangle, \cdots |e_n\rangle$ of vectors is said to be orthonormal if

$$\langle e_i|e_j\rangle = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.4}$$

- An important consequence of the axioms defining the scalar product is the *Schwarz inequality*

$$\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq \langle\phi|\psi\rangle\langle\psi|\phi\rangle, \tag{1.5}$$

where the equality holds if and only if the two vectors in question are *linearly dependent* i.e. if

$$|\psi\rangle = \mu|\phi\rangle, \tag{1.6}$$

μ being a complex number. In order to establish this, show that the minimum value of $\langle\Psi(\mu)|\Psi(\mu)\rangle$, where $|\Psi\rangle = |\psi\rangle - \mu|\phi\rangle$, as a function of μ is $\langle\psi|\psi\rangle - |\langle\psi|\phi\rangle|^2/\langle\phi|\phi\rangle$. The requirement that this value, due to axiom 3 of the scalar product, be positive leads to the Schwarz inequality in (1.5). Also, according to the axiom 4 above, $\langle\Psi(\mu)|\Psi(\mu)\rangle = 0$ iff $|\Psi(\mu)\rangle = 0$ i.e.

iff (1.6) holds. It may be verified easily that (1.5) then holds with equality. In a similar way we can derive the generalized Schwarz inequality

$$\det(\langle\psi_\mu|\psi_\nu\rangle) \geq 0, \tag{1.7}$$

where $\det(\langle\psi_\mu|\psi_\nu\rangle)$ is the determinant of the matrix constituted by the elements $\langle\psi_\mu|\psi_\nu\rangle$, $\mu, \nu = 1, \ldots, n$. Invoking the fact that the determinant of a matrix is zero if its rows (or columns) are linearly dependent, it follows that the equality in (1.7) holds iff $|\psi_\mu\rangle$ are linearly dependent.

Completeness. In a scalar product vector space of finite dimension n, there always exists a set of n linearly independent vectors $\{|\psi_i\rangle\}$, called the *basis* vectors, such that any vector $|\psi\rangle$ can be expressed as a linear combination [2],

$$|\psi\rangle = \sum_{i=1}^{n} d_i|\psi_i\rangle. \tag{1.8}$$

The complex numbers $\{d_i\}$ in a scalar product space may be determined by taking the scalar product of (1.8) with the vectors $\{|\phi_i\rangle\}$ orthonormal to $\{|\psi_i\rangle\}$ to give $d_i = \langle\phi_i|\psi\rangle$ so that

$$|\psi\rangle = \sum_{i=1}^{n} |\psi_i\rangle\langle\phi_i|\psi\rangle. \tag{1.9}$$

The vector $|\psi\rangle$ in an n-dimensional space is thus characterized by n complex numbers $\{\langle\phi_i|\psi\rangle\}$. The column of these numbers constitutes a *representation* of the vector in the given basis. The dual $\langle\psi|$ of $|\psi\rangle$ is then represented by the row constituted by the numbers $\{\langle\psi|\phi_i\rangle\} = \{\langle\phi_i|\psi\rangle^*\}$. Thus the representation of $\langle\psi|$ is obtained by the process of *hermitian conjugation* (interchanging of rows and columns along with the operation of complex conjugation), denoted by $\dagger$, of $|\psi\rangle$: $\langle\psi| = (|\psi\rangle)^\dagger$.

The expansion (1.9) in a scalar product space is guaranteed if the space is finite-dimensional. However, such an expansion need not exist if the space is infinite-dimensional. In quantum mechanics, we are concerned only with those scalar product linear vector spaces in which every vector is expressible in terms of a basis. Such a space is called a Hilbert space.

Now, on invoking the fact that (1.9) is to be valid for an arbitrary $|\psi\rangle$, follows the *completeness relation*

$$\sum_{i=1}^{n} |\psi_i\rangle\langle\phi_i| = I, \tag{1.10}$$

where I is the identity operator defined below. If $\{|\psi_i\rangle\}$ is orthonormal, i.e. if $\{|\phi_i\rangle\} = \{|\psi_i\rangle\} \equiv \{|e_i\rangle\}$, then (1.10) reduces to

$$\sum_{i=1}^{n} |e_i\rangle\langle e_i| = I. \tag{1.11}$$

In our discussion so far we have assumed that the basis vectors are denumerable. There are, however, occasions which require us to work with a basis labeled by a continuous parameter. Consider an orthonormal set of basis vectors $|\xi\rangle$ labeled by a real continuous parameter ξ. The condition of orthonormality then reads

$$\langle\xi'|\xi\rangle = \delta(\xi - \xi'), \tag{1.12}$$

$\delta(x)$ being the Dirac delta function. If $a \le \xi \le b$ then the analog of the expansion of a vector in terms of the basis vectors is

$$|\psi\rangle = \int_a^b \langle\xi|\psi\rangle\ |\xi\rangle\ \mathrm{d}\xi. \tag{1.13}$$

A vector $|\psi\rangle$ in a continuous basis is thus represented by the function $c(\xi) = \langle\xi|\psi\rangle$ of a real variable ξ.

Operators. The action of a force transforms the state of a system. A transformation of a state of a system may be described by a rule, called an *operator*, that associates with a vector in the space another vector in the same space. If, for example, an action transforms $|\psi\rangle$ to $|\phi\rangle$ then we write

$$\hat{A}|\psi\rangle = |\phi\rangle \tag{1.14}$$

where the operator $\hat{A}$ defines the rule of transformation. We distinguish an operator from a c-number variable by a caret on the former. An operator $\hat{A}$ is *linear* if, for any complex numbers c_1 and c_2,

$$\hat{A}\Big(c_1|\psi\rangle + c_2|\phi\rangle\Big) = \Big(c_1\hat{A}|\psi\rangle + c_2\hat{A}|\phi\rangle\Big). \tag{1.15}$$

We shall be concerned only with linear operators.

If $\hat{A}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$ then $\hat{A}$ is called the *unit* or *identity* operator, often denoted by I. Since I acts like the scalar unity, we do not dress it with a caret and even denote it by 1.

In order to obtain a c-number representation of an operator $\hat{A}$, consider an orthonormal basis $\{|e_i\rangle\}$. Rewrite $\hat{A}$ as $I\hat{A}I$ where I is the unit operator and express I in terms the completeness relation (1.11) to get

$$\hat{A} = \sum_{i,j=1}^{n} \langle e_i|\hat{A}|e_j\rangle\ |e_i\rangle\langle e_j|. \tag{1.16}$$

The operator $\hat{A}$ may be represented by an $n \times n$ matrix constituted by the complex numbers $\langle e_i|\hat{A}|e_j\rangle$ $(i, j = 1, \ldots, n)$. On operating (1.16) on an arbitrary vector $|\psi\rangle$, it follows that $\hat{A}|\psi\rangle$ is represented by the product of the matrix $\langle e_i|\hat{A}|e_j\rangle$ representing $\hat{A}$ with the column $\langle e_j|\psi\rangle$ representing $|\psi\rangle$. It is straightforward also to show that a product $\hat{A}\hat{B}$ is represented by the product of the matrices representing them. Thus, the correspondence between vectors as columns and operators as matrices is not only notational but also operational.