INTRODUCTION TO
REAL ANALYSIS

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Preface

This text is an introductory course in real analysis, intended for students who have completed a calculus sequence. It is also designed to serve as preparation for advanced mathematics courses of many sorts. Though there are occasional references to it in exercises, linear algebra is not specifically a prerequisite for this text. Nevertheless, the changing role of linear algebra in the undergraduate curriculum is one of the main reasons this book comes to be the way it is. In the past, a first course in linear algebra was generally considered to be the place where one "learned to do proofs." The mathematics curriculum has gradually changed, though, and proofs as such are no longer the main focus of the typical linear algebra course. As a result, a student's first extensive experience with the logical and organizational skills necessary for the successful construction of proofs is often delayed until they find themselves in courses in which success is predicated on possession of those very skills.

Textbooks have been slow to adapt to these changes. This book provides a pathway from the calculus course to real analysis (and beyond) in which the discussion of the construction of proofs is a continuing and central theme. Throughout the text (but most especially in Part 2), proofs are not simply presented in final form. Rather they are shown as works-in-progress; they are *built*, and their construction is discussed along with their final content. While learning real analysis, the student will, it is hoped, also learn something about the workings of the mind of a mathematician—invaluable information if one is to become a researcher in or a teacher of mathematics, or both, but information that is often overwhelmed by the demands of the subject at hand.

The text is organized into four parts. The material of Part 1 is the common foundation of most upper-level mathematics courses. The book begins with an introduction to basic logical structures and techniques of proof. The ideas introduced here, especially the crucial "forward-backward method" of constructing proofs, are all emphasized and used explicitly throughout the text. The rest of Part 1 includes discussions of the concept of cardinality, the algebraic and order structures of the real and rational number systems, and the natural numbers in their dual role as the basis for induction and as special elements in ordered fields. The discussion of the real and rational number systems sets the stage for

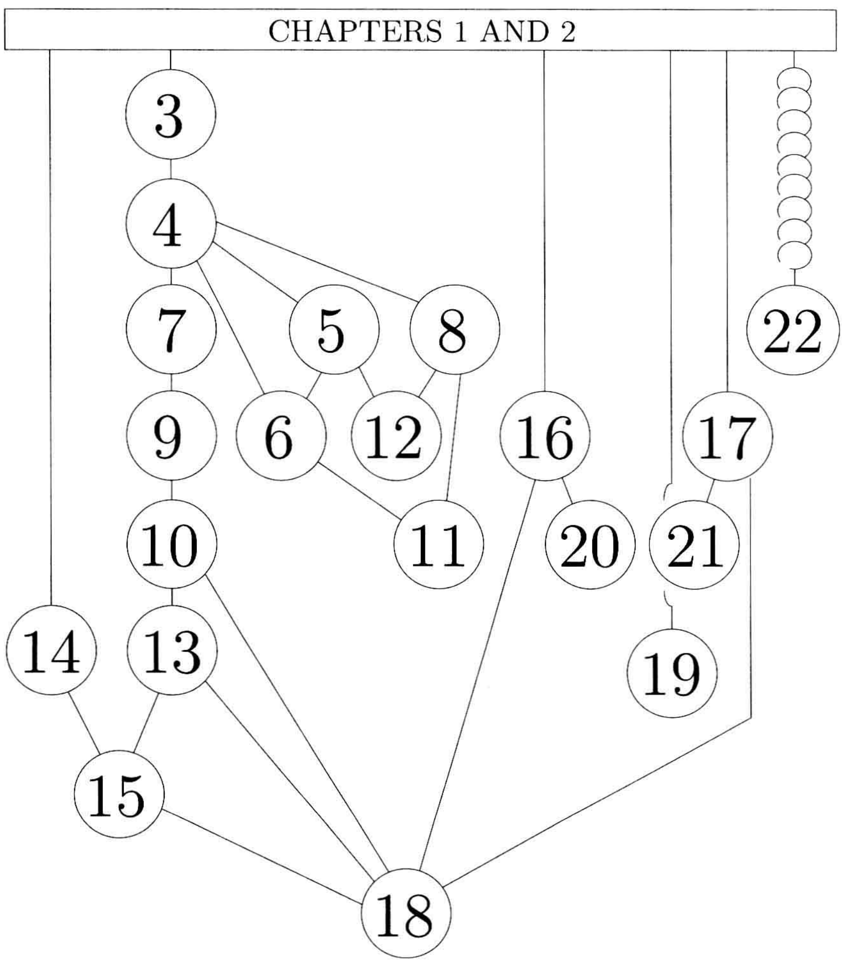
Part 2 of the book. In Part 1 it is found that these systems have much in common, while Part 2 is devoted to discovering how they are different.

Part 2 of the text is an in-depth examination of the completeness of the real number system in its various guises and a discussion of the topological structure of the real number system. It is here also that the discussion of the construction of proofs becomes most focused. The student is regularly reminded of the relationship between the topic at hand and the larger story by a graphic device called the "Big Picture." The Big Picture, which describes interrelationships among the various manifestations of the completeness of the real number system, gives a structure and unity to the subject not found in most mathematics courses. The organization of Part 2 is such that after its first four chapters are discussed, the rest may be treated in any order and at the discretion of the instructor. The structure provided by the Big Picture allows the option of showing that the important properties of the real number line do not exist in isolation; they are in fact equivalent.

Part 3 of the text is a review and extension of calculus in light of the student's new understanding of the real number system. With the knowledge and experience gained in Part 2, the student can appreciate the structures and results of calculus to a depth not possible in a first-year course, and is prepared to deal with proofs presented in a more direct and traditional manner. Part 4 is a selection of topics in real function theory, investigated as natural outgrowths of questions readily understood by the student. The instructor has great latitude to choose topics in the second half of the text, and coverage of this material can range from an intensive reconstruction of calculus to a discussion approaching current research topics.

The exercises in the text are many and varied. They range from the fairly routine—checking steps omitted from examples and observing mechanical results at work—to completion of partial proofs from the text to extended projects, some of which border on current research. There are questions in which the student is asked simply to discuss a statement, or to explain to their own satisfaction why something is true or false, free from the restrictions of a formal proof. There are questions in which the student is asked to find flaws in incorrect (though possibly convincing) "proofs," and questions in which the student is asked to reconsider their own proofs in light of new information. Most importantly, the exercises are an integral part of the text. Regular and meaningful cross-referencing reminds the student of the unity of the subject and highlights their own active role in its development. Furthermore, the exercises themselves constitute part of the ongoing study of the workings of the mathematical mind, as the student is often led from one topic to another in a way that suggests, it is hoped, that no matter how many answers one finds, there are always more questions to be asked.

The diagram below may be interpreted like this: The overall flow of the subject is from top to bottom. The material of Chapter 1 supports everything else, and the ideas in Chapter 2 pop up just about everywhere. Strong dependence between chapters is indicated by a line, though ideas from a chapter represented by a higher ball might be needed in one below it (spatially or structurally). For instance, it is necessary to understand the material of Chapter 4 to make sense of Chapter 5, but Chapter 4 is less directly needed in Chapter 12 and even less so in Chapter 20. Aside from experience, there are no prerequisites for Chapters 14 and 19, and Chapter 22 may be taken up any time after Chapter 5 (hence its variable position). Part 2 of the book (Chapters 5 through 12) has an internal structure of its own, which is described in the introduction to that section.



Gratitude, like proofs, should sometimes follow the forward-backward method. I am eternally grateful to Professor Ronald Shonkwiler of the Georgia Institute of Technology for starting me on my way to becoming a mathematician, and to Professor Daniel Waterman and the late David Williams of Syracuse University for doing their best to help me finish the journey. In the present, I am indebted to Jim Spencer of the University of South Carolina at Spartanburg and Robert E. Zink of Purdue University, and others whose identities I will never know, for their careful and thoughtful reviews of the manuscript of this book. Their suggestions have much improved the end result. For the future, I wish to thank the students who have been so tolerant during the development of this text. They have, with the unerring radar available only to those to whom a subject is new, corrected shocking slews of errors, and have, for the most part, helped me overcome my tendency to write questions in which it is necessary to do part (b) before part (a). They have also, much as was hoped, asked lots of questions, and many of the exercises in the text were proposed by students in the course. It can be said with equal validity in regard to all three of these groups of people, that most of what is good here is theirs, while the remaining errors and oversights are mine alone.

Contents

PART THREE: TOPICS FROM CALCULUS

Part One

Preliminaries

We begin our work with a discussion of the construction of proofs. Writing proofs, of course, is the heart of mathematics. If there were always direct, mechanical processes for writing proofs, though, mathematics would not be nearly so fascinating. We will find that proofs need not be as mysterious as they might seem and that we can smooth the way considerably by making use of some basic organizational schemes.

The rest of the first half of the book is best understood by thinking of what we will call the **Big Question**: How are the real numbers different from the Rational Numbers? We will tackle this in two stages. In Part 1 of the book we will see some of the ways these two number systems are similar. It's best to know the ways things are alike before asking how they are different. In Part 2 of the book, we examine the differences between them and answer the Big Question. Our efforts will be richly rewarded. We will find that the property that distinguishes the real number system from the rational number system is precisely what makes calculus work.

Through all of this we will never say what the real numbers actually are! That we can consider working this way is one of the remarkable features of mathematics. We can study how the real numbers *work*, blissfully unconcerned with what they *are*. We can solve the crime, so to speak, without ever knowing the suspects. We will finally meet the real numbers at the very end of the book. Agatha Christie would be proud.[1] If we don't even know what they are, how can we hope to say that the real numbers are different from the rational numbers? Like this: If an object X possesses some mathematical property that the object Y does not, we can say with confidence that X is different from Y. If X is definitely red, Y is definitely blue, and (this is most important) a red thing can't also be blue, then X and Y must be different. This sort of argument underlies much of what happens in this book. Be sure to watch for it.

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[1] Agatha Christie annoyed faithful readers of her wonderful mysteries for decades by revealing essential clues only in the final scene ("I suppose you're wondering why I've called you all here ...").

Chapter 1
Building Proofs

1.1 A QUEST FOR CERTAINTY

The study of mathematics is the quest for a sort of certainty that can be attained in no other endeavor. In mathematics we can "prove" things. But what does this mean? Less than we might hope. Bertrand Russell, one of the foremost British philosophers of recent times, called mathematics "the subject in which we never know what we are talking about, nor whether what we are saying is true." If this is the case, how can we hope to prove anything? We can't! What we *can* do, however, is show with absolute certainty that each of a chain of statements is "as true as those before it." If we believe the statements at the beginning of the chain, and that the chain is properly assembled, we *must* believe the statements at the end.[1]

Of course, we have to begin somewhere, and it is evident that statements at the beginning of such a chain *can't* be proved. Statements that we agree to accept without proof are called **axioms**. We may discuss whether an axiom is appropriate (that is, whether it describes life as we perceive it) and we might at some point want to spend time discussing which axioms we ought to believe and which we should reject. But once this issue has been settled (and for the purposes of this course we consider it to be so) we agree *not* to discuss whether an axiom is true or false. Though it certainly can be an activity of great value, it is not our goal to scrutinize a collection of axioms here. We are studying the "top floors" of a subject, not its "foundations." Besides, the chain of reasoning leading from the most basic axioms to this text is unimaginably long. Bertrand Russell and Alfred North Whitehead took it upon themselves to build such a chain in their monumental *Principia Mathematica*. After several *hundred* pages, they were able to prove from "first principles"

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[1] This is the price mathematicians pay for the power to make proofs. In mathematics we find things that we are *compelled* to believe (some of which we might prefer not to). Practitioners of other fields have more flexibility to pick and choose what they will accept.

that $1 + 1 = 2$. Fortunately, this is not how we will be spending our time. Foundational questions are as much philosophical as mathematical, and as mathematics they fall under headings other than analysis.

EXERCISES 1.1

1. What did Russell mean when he said that mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true"?

2. Discuss the differences in meaning of a statement of the form "It is true" when the assertion is made by a mathematician, a physicist, a biologist, a sociologist, a politician, or a used-car salesperson.

1.2 PROOFS AS CHAINS

It is instructive in many ways to view a proof as a chain of reasoning. To build a chain, we need a supply of links and a way to connect one link to another. In geometry class, we sometimes made two-column proofs with "Steps" on one side of the paper and "Reasons" on the other. A step might have been "Angle a is congruent to angle b," with the reason "Alternate interior angles." Steps are the links in the chain; reasons are the connections between them. We can safely use a real chain only if each of its links and the connections between them are sound. In mathematics, a sequence of statements, each of which is properly formed and correctly justified by those before it, is called a **proof**.

We can construct a chain from either end, or from both ends at once. We can even assemble links bound for the middle into sections and then connect the sections. In the same way, we can work a proof from the beginning or the end (or even from the middle). A proof is almost never thought of straight through from beginning to end. In textbooks, though, proofs are usually *written down* from beginning to end, causing much unnecessary confusion. Here we will give a brief outline of the basic logical structures we will encounter in our work. We observe right off the bat that even the simplest of ideas sometimes warrant discussion.

1.3 STATEMENTS

We have already used this important word, even though we may not be entirely sure what it means. Since statements are the steps in our proofs—the links in our chains—we should examine the meaning of the