CALCULUS with ANALYTIC GEOMETRY

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Preface

This is a textbook on calculus and analytic geometry that is designed for (a) the prospective mathematics major, (b) a student interested in the physical sciences, (c) engineering students, or (d) anyone who simply wishes to develop an understanding of the concepts and applications of calculus. The author was guided by the following objectives:

(i) The student should be able to read this textbook and learn from it. A textbook must be more than a collection of theorems, examples, and exercises.

(ii) Definitions, theorems, and their proofs must be given in a straightforward manner with rigor that is appropriate for a beginner in this field. At the same time, this text is not written in a cookbook style because this would be an obvious disservice to the prospective reader.

(iii) The illustrative examples should be ample in number, properly ordered, easy to comprehend, and a definite asset in the solution of many of the exercises. A strong effort has been made to see that this correlation does exist.

(iv) The exercises should be sufficient in number, properly ordered, and generally graded as to order of difficulty. The author has solved each of the exercises so that he hopes there will be a minimum of numerical difficulties.

Chapter 1 starts with some basic facts about the real number system. Rational and irrational numbers and their decimal representations are discussed briefly. Particular attention is then given to the rules necessary for the manipulation of inequalities. Next, the solution of equalities and inequalities involving the absolute value is developed. The major portion of this chapter is devoted to the elements of plane analytic geometry involving studies of the straight line, the circle, and the parabola. The analysis of intercepts, symmetry, extent, and vertical and horizontal asymptotes, with applications to curve sketching, is presented in detail. In the last section of this chapter, there is a brief treatment of mathematical induction because of its application later in the text.

Chapter 2 is devoted to the concepts of (i) function, (ii) limit of a function, and (iii) continuity. The concept of function is defined in terms of rule of correspondence and then ordered pairs. The main thrust of this chapter is the development of limit of a function, given in Sections 2.4 and 2.5. The former section treats this material intuitively, avoiding the ϵ-δ concept. This was purposely done to make it simple for an instructor to bypass the ϵ-δ method (if desired) by skipping Section 2.5. Those instructors who want to present the ϵ-δ method can proceed with the intuitive discussion (or omit it altogether) and emphasize Section 2.5.

The concept of the derivative and the rules for differentiation are the primary subjects of Chapter 3. Introductory material on the tangent to a curve and velocity are given to motivate the definition of the derivative. An intuitive (heuristic) development of the chain rule is followed by a rigorous treatment. A brief historical introduction to calculus is given in the first section of this chapter. The author hopes that this material will be read and reread as maturity and technical know-how develop. Chapter 4 is devoted to applications of the derivative to problems of related rates, maxima and minima, curve sketching, and problems in economics.

In Chapter 5 the concepts of differentials and antiderivatives are treated where the standard notation for antiderivatives, $\int f(x)\,dx$, is used. The techniques of substitution and the generalized power rule are developed and the chapter closes with applications to elementary initial value and boundary value problems.

Summation notation is treated in the first section of Chapter 6 and is then applied to the concept of area under a curve as a limit of sums. This motivates the definition of the Riemann integral as a limit of a summation. However, as we are careful to point out, area is just one of the properties of the definite integral. The properties of the definite integral are then developed, leading to the fundamental theorems of calculus. These theorems bind together the two apparently (and historically) diverse ideas of differentiation and the definite integral.

Some of the numerous applications of the definite integral are investigated in Chapter 7. The objective is to explain each of these concepts in a lucid, straightforward, and interesting manner. The concept of element is the unifying idea that threads its way through this material. Hence we have an element of area, an element of volume, an element of work, and so on. These then yield the definite integral that pertains to the given application.

Chapter 8 begins with the modern definition of the natural logarithmic function by means of a definite integral. From this the properties of the logarithmic function are systematically deduced and application of the process known as logarithmic differentiation is given. Integrals yielding logarithmic functions are then considered. At this juncture, the Euler constant e is defined and interpreted geometrically, and relations that yield this number are developed briefly. The role of e in the case of continuous compound interest follows. Then the concept of functions and their inverses leads to the introduction of the exponential function and its useful properties. Applications of the exponential function are made to problems involving variations in atmospheric pressure, growth, and decay.

The first section of Chapter 9 constitutes a rather detailed review of trigonometry, necessary because of the wide variation of student preparation in this discipline. The other seven sections develop the differentiation and integration of the trigonometric, inverse trigonometric, and hyperbolic functions. Applications of these functions to rate problems, periodic motion, the analysis of a particle moving through a resistive medium, and the determination of deflection and tension in a hanging cable are given here. The usefulness of these functions in the evaluation of integrals is demonstrated in a number of instances.

The primary objective of Chapter 10 is to develop the important methods of formal integration. We start with the integration of trigonometric and algebraic functions by applying certain substitutions and the generalized power formula. The integration of rational functions is motivated and then treated in detail. It is followed by the very useful procedure of integration by parts. In Section 10.10 the practical use of integral tables is demonstrated. The last two sections are concerned with the derivation of the trapezoidal and Simpson's rules for the numerical integration of definite integrals.

Vectors in the plane is the subject of Chapter 11. After a brief historical introduction, vector algebra is developed and applied to problems in physics and geometry. Next, we investigate vector functions and the parametric representation of plane curves. This enables us to examine some of the more interesting plane curves such as cycloids. Vector calculus is introduced and applications are given to motion in two dimensions, where concepts such as velocity, acceleration, arc length, and curvature are treated.

In Chapter 12 the reader is introduced to plane polar coordinates. These coordinates are related to the Cartesian coordinates, and equations of simple plane curves are expressed in both coordinate systems. Various tests for symmetry of curves expressed in polar coordinates are derived. After this we consider the intersection of plane curves given in polar coordinates. This is a more intricate problem in polar coordinates than in rectangular coordinates because there is no longer a one-to-one correspondence between points and their polar coordinate representation. Then the formula for the length of a curve in polar coordinates is easily derived from the expression in rectangular coordinates. The chapter closes with problems of motion in the plane using vector calculus in the derivation of the velocity and acceleration components in polar coordinates.

Chapter 13 first presents a brief historical introduction illustrating the reason for the name "conic sections." In the next two sections the ellipse and hyperbola are treated in a traditional manner. Then the three conic sections—the ellipse, the hyperbola, and the parabola—are unified by using the concepts of focus, directrix, and eccentricity. Rotation of axes and the utility of both rotation and translation of axes in the simplification of conic sections in nonstandard form are discussed. In the last section, the polar coordinate representations of conic sections are developed.

The topics of the evaluation of indeterminate forms by L'Hospital's rules, improper integrals, the extended law of the mean (Taylor's formula), and the Newton-Raphson method for root determination constitute the subject matter of Chapter 14.

Chapter 15 gives a fairly comprehensive treatment of infinite sequences and infinite series. The ratio, root, integral, comparison, and Leibnitz tests are described. Absolute and conditional convergence and the question of rearrangement of terms of an infinite series are discussed. Also there is a development of interval of convergence for power series, and term by term differentiation and integration of power series are treated. Applications are given for the use of Taylor and Maclaurin series in the representation of functions, the evaluation of integrals, and the solution of elementary initial value problems.

The application of vectors in three-dimensional space to problems in solid analytic geometry is given in Chapter 16. The dot and cross products are defined and applications to the line in space and the plane are developed. Cylinders, surfaces of revolution, and quadratic surfaces are also discussed. Vector functions and their derivatives are applied to curves in space. Additional applications of vectors in space are given to velocity, acceleration, curvature, and the rudiments of differential geometry.

Chapter 17 starts with the concept of functions of two or more variables. Then we proceed to partial derivatives of first and higher orders, limits, continuity, differentiability, differentials, and the chain rule. The concepts of the directional derivative, gradient, and the tangent plane and the normal line to a surface are developed. The chapter closes with a discussion of the absolute maximum and minimum problem for a function of two variables, the method of least squares, and an introduction to the Lagrange multiplier method.

Double and triple integration and their applications to problems such as the determination of volumes, moments, center of mass, surface area, and moments of inertia are developed in Chapter 18.

Appendix A includes review formulas from algebra, plane geometry, trigonometry, and solid geometry.

Appendix B gives nine tables of numerical data associated with particular functions such as squares, cubes, the natural logarithmic function, the exponential function, trigonometric functions (radians and degrees), hyperbolic functions, and common logarithms.

Inside the front and back covers there is a table of derivatives and integrals. Of course, the table of integrals should be used *after* the reader is familiar with the formal integration process as discussed in Chapter 10.

A number of my colleagues aided me in achieving the final form for the first edition. Professors Joel Stemple, Robert McGuigan, James E. Anderson, James M. Stakkestad, and Richard Crownover served as reviewers of the manuscript. I wish to thank them for their effective and generous assistance in this project. Furthermore, Joel Stemple helped me revise my original manuscript in accordance with the comments of *all* the reviewers and, through our mutual interaction, a better book resulted. In addition, my thanks go to Professor Nathaniel R. Riesenberg and Stephanie M. White for their critical reading of the manuscript and their valuable suggestions for its improvement.

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Contents

1

Real Numbers and Elementary Analytic Geometry

1.1 Real Numbers, Inequalities, and Intervals

Most aspects of calculus and analytic geometry involve real numbers. We have considerable experience with the operations of addition, subtraction, multiplication, and division and, therefore, we shall not list the axioms that explain these manipulations of real numbers. We also shall not state definitions that are by now familiar to all of us. Instead, our attention will be restricted to a few particularly important definitions and rules that will be used in our development of calculus.

For convenience, we use the symbol R_1 to denote the set of all real numbers. Unless stated otherwise, we shall assume that all numbers we work with are members of R_1.

The real numbers can be broken into two parts—"rational numbers" and "irrational numbers." We define these as follows:

Definition A ***rational number*** is any number that can be expressed as a quotient of integers (that is, a rational number can be put in the form $\frac{a}{d}$ where a and d are integers and $d \neq 0$).

Examples of rational numbers are

$$\frac{3}{5} \qquad -\frac{1}{2} \qquad \frac{25}{8} \qquad \frac{14}{7} \qquad -21 \qquad \frac{9\sqrt{3}}{2\sqrt{3}} \qquad \frac{\sqrt{24}}{\sqrt{6}}$$

To see that $\frac{\sqrt{24}}{\sqrt{6}}$ is rational we note that

$$\frac{\sqrt{24}}{\sqrt{6}} = \frac{\sqrt{(4)(6)}}{\sqrt{6}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{6}} = \sqrt{4} = 2 = \frac{2}{1}$$

Definition An ***irrational number*** is any real number that is not rational.

Examples of irrational numbers are

$$\sqrt{2} \qquad -\sqrt{7} \qquad \pi \qquad \frac{3\sqrt{5}}{2} \qquad \frac{\sqrt{6}}{\sqrt{2}}$$

$\left(\text{Note that } \frac{\sqrt{6}}{\sqrt{2}} = \frac{\sqrt{(3)(2)}}{\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{2}} = \sqrt{3}\right)$

It can be shown that a number is rational if and only if it can be expressed as either a "terminating decimal" or a "repeating decimal." For example

$$\left.\begin{aligned} \frac{5}{4} &= 1.25 \\ \frac{-127}{10000} &= -0.0127 \end{aligned}\right\} \quad \text{terminating decimals}$$

$$\left.\begin{aligned} \frac{13}{6} &= 2.166666\cdots \\ \frac{1}{99} &= 0.010101\cdots \end{aligned}\right\} \quad \text{repeating decimals}$$

(the symbolism $\cdots$ means that the indicated pattern continues).

In a repeating decimal, the terms do not have to start repeating immediately—in $\frac{13}{6} = 2.16666\cdots$, only after we reach the first 6 do the numbers repeat.

The decimal expansion for irrational numbers is neither terminating nor repeating. There is no pattern to the expansion and the expansion never ends.

Definition We say that ***a is less than b*** and ***b is greater than a*** if and only if $b - a$ is a positive number.

$a < b$ is the notation for "a is less than b"
$b > a$ is the notation for "b is greater than a"

We call $a < b$ and $b > a$ ***strict inequalities.*** If we wish to allow for the possibility that a and b might also be equal, we have the ***nonstrict inequalities*** defined by

$a \leq b$ if and only if either $a < b$ or $a = b$
$b \geq a$ if and only if either $b > a$ or $b = a$

Some numerical examples of inequalities are $3 < 6$, $8 > 2$, $4 \leq 5$, $-1 > -2$, $7 \geq 7$, $6 \geq -5$.

The following are rules that may be used when working with inequalities:

1. (A) if $a < b$ and $b < c$ then $a < c$ } the ***transitive law***
   (B) if $a > b$ and $b > c$ then $a > c$ }
2. (A) if $a < b$ then $a + c < b + c$
   (B) if $a > b$ then $a + c > b + c$
3. (A) if $a < b$ then $a - c < b - c$
   (B) if $a > b$ then $a - c > b - c$

4. (A) if $a < b$ and $c > 0$ then $ac < bc$
   (B) if $a > b$ and $c > 0$ then $ac > bc$
5. (A) if $a < b$ and $c < 0$ then $ac > bc$
   (B) if $a > b$ and $c < 0$ then $ac < bc$
6. (A) if $0 < a < b$ and $0 < c < d$ then $0 < ac < bd$
   (B) if $a > b > 0$ and $c > d > 0$ then $ac > bd > 0$

Rules (2) and (3) say that you can add a number to or subtract a number from both sides of an inequality without changing the inequality sign; (4) and (5) say that, if both sides of an inequality are multiplied by the same number (not zero), then the inequality sign is the same if the number is positive and the inequality sign reverses if the number is negative. These rules can all be proven without much difficulty. However, the reader might find it more instructive first to construct some numerical examples to illustrate each of the rules.

We have already used the phrase "if and only if" (sometimes abbreviated "iff") several times. This phrase says that the two expressions it connects are equivalent; that is, a statement and its converse are simultaneously both true or both false. Thus, for example, the statement "$a < b$ if and only if $b - a$ is positive" means that "if $a < b$ then $b - a$ is positive" and also that "if $b - a$ is positive then $a < b$."

In the usual manner, we represent the real numbers R_1 geometrically by points on a horizontal line, which may be extended indefinitely (see Figure 1.1.1). An arbitrary point on the horizontal line (also called the axis) is chosen to represent the number 0. This point is taken to be the ***origin*** of coordinates. An arbitrary unit of distance is chosen. Then each positive number x is represented by a point x units to the right of the origin. Similarly, each negative number x is represented by a point at a distance $-x$ units to the left of the origin. Furthermore, it is postulated that there is a one-to-one correspondence between R_1 and the number line. This means that to each point on the number line there corresponds precisely one real number and, conversely, to each real number there is one and only one point on the line. Then two *distinct* points P and Q on the number line cannot represent the same number. Therefore the points on the number line may be identified with the numbers represented by them.

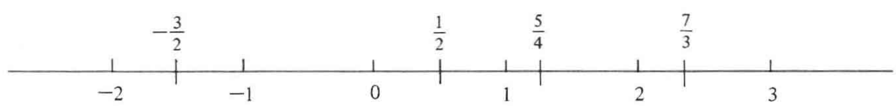

Figure 1.1.1

By our convention, $a < b$ if and only if the point representing the number a is to the left of the point representing the number b. Similarly, $c > d$ if and only if the point representing the number c is to the right of the point representing the number d. Thus, for example, the point representing $-\frac{3}{2}$ is to the left of the point representing -1. The point representing 2 is to the right of the point representing $\frac{1}{2}$.

A number x is said to be between the numbers a and b if and only if $a < x$ and $x < b$. We write this as a continued inequality as follows

$$a < x < b \tag{1}$$

The continued inequality (1) is called an ***open interval*** and is denoted (a, b). The set of all points x such that $a \leq x \leq b$ is called a ***closed interval*** and is denoted [a, b]. When talking about intervals, "closed" means the endpoints are included whereas "open" means they are excluded. The open and closed intervals are shown in Figure 1.1.2.

(a) Open interval (b) Closed interval

Figure 1.1.2

The interval $a < x \leq b$ is said to be ***half-open on the left*** and is denoted by $(a, b]$. Also, the interval $a \leq x < b$ is ***half-open on the right*** and is denoted by $[a, b)$ (Figure 1.1.3). In all cases $b - a$ is called the ***length*** of the interval and all such intervals are finite.

(a) Half open interval on the left (b) Half open interval on the right

Figure 1.1.3

To denote the nonfinite or infinite intervals, symbols such as $-\infty$ and ∞ will be used. The reader must be cautioned not to confuse these symbols with real numbers because they do not obey the properties of real numbers. The set of all x such that $x > a$ is called the infinite interval (a, ∞). Similarly, $[a, \infty)$ is the set of all x with $x \geq a$, $(-\infty, b)$ the set of all $x < b$ and $(-\infty, b]$ the set of all x with $x \leq b$ (Figure 1.1.4). Finally $(-\infty, \infty)$ is the set of all real x and is represented by the whole number line.

Figure 1.1.4

Let us now turn to the solution of inequalities.

EXAMPLE 1 Find all real numbers satisfying the inequality $-3x > 6$.

Solution Division of both sides of the given inequality by -3 and reversing the sense of the inequality yields

$$x < -2$$

Therefore $-3x > 6$ if and only if $x < -2$. The solution interval is $(-\infty, -2)$ and is shown in Figure 1.1.5. ●

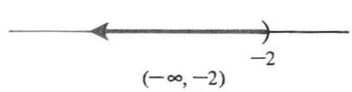

Figure 1.1.5