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To the memory of my mother,
Nadejda Vassilievna Pajitnova

Contents

Preface

In the early 1920s M. Morse discovered that the number of critical points of a smooth function on a manifold is closely related to the topology of the manifold. This became a starting point of *the Morse theory* which is now one of the basic parts of differential topology. Reformulated in modern terms, the geometric essence of Morse theory is as follows. For a C^∞ function on a closed manifold having only non-degenerate critical points (*a Morse function*) there is a chain complex $\mathcal{M}_*$ (the *Morse complex*) freely generated by the set of all critical points of f, such that the homology of $\mathcal{M}_*$ is isomorphic to the homology of the manifold. The boundary operators in this complex are related to the geometry of the gradient flow of the function.

It is natural to consider also *circle-valued Morse functions*, that is, C^∞ functions with values in S^1 having only non-degenerate critical points. The study of such functions was initiated by S. P. Novikov in the early 1980s in relation to a problem in hydrodynamics. The formulation of the circle-valued Morse theory as a new branch of topology with its own problems and goals was outlined in Novikov's papers [**102**], [**105**].

At present the Morse-Novikov theory is a large and actively developing domain of differential topology, with applications and connections to many geometrical problems. Without aiming at an exhaustive list, let us mention here applications to the Arnol'd Conjecture in the theory of Lagrangian intersections, fibrations of manifolds over the circle, dynamical zeta functions, and the theory of knots and links in S^3. The aim of the present book is to give a systematic treatment of the geometric foundations of the subject and of some recent research results.

The central geometrical construction of the circle-valued Morse theory is *the Novikov complex*, introduced by Novikov in [**102**]. It is a generalization to the circle-valued case of its classical predecessor — the Morse complex. Our approach to the subject is based on this construction.

We begin with a detailed account of several topics of the classical Morse theory with a special emphasis on the Morse complex. Part 1 is introductory: we discuss Morse functions and their gradients. The contents of the first chapter of Part 2 is the Kupka-Smale Transversality theory; then we define and study the Morse complex.

In Part 3 we discuss the notion of *cellular gradients* of Morse functions, introduced in the author's papers [**113**], [**108**]. To explain the basic idea, we recall that for a Morse function $f : W \to [a, b]$ on a cobordism W the gradient descent determines a map (not everywhere defined) from the upper boundary $f^{-1}(b)$ to the lower boundary $f^{-1}(a)$. It turns out that for a C^0-generic gradient this map can be endowed with a structure closely resembling the structure of a cellular map. We work in this part only with real-valued Morse functions, however the motivation comes from later applications to the circle-valued Morse theory.

In Part 4 we proceed to circle-valued Morse functions. In Chapter 11 we define the Novikov complex. Similarly to the Morse complex of a real-valued function, the Novikov complex of a circle-valued Morse function is a chain complex of free modules generated by the critical points of the function. The difference is that the base ring of the Novikov complex is no longer the ring of integers, but the ring $\overline{L}$ of Laurent series in one variable with integral coefficients and finite negative part. The homology of the Novikov complex can be interpreted as the homology of the underlying manifold with suitable local coefficients.

The boundary operators in the Novikov complex are represented by matrices with coefficients in $\overline{L}$ (the Novikov incidence coefficients). One basic direction of research in the Morse-Novikov theory is to understand the properties of these Laurent series. The Novikov exponential growth conjecture says that these series always have a non-zero radius of convergence. A theorem due to the author (1995) asserts that for a C^0-generic gradient v of a circle-valued Morse function, every Novikov incidence coefficient is the Taylor series of a rational function. This theorem is the basis for the contents of Chapter 12. The reader will note that in general we emphasize *the C^0 topology* in the space of C^∞ vector fields; we believe that it is the natural framework for studying the Morse and Novikov complexes.

These results are then applied in Chapter 13 to the dynamics of the gradient flow of the circle-valued Morse functions. We obtain a formula which expresses the Lefschetz zeta functions of the gradient flow in terms of the homotopy invariants of the Novikov complex and the underlying manifold.

The last chapter of the book contains a survey of some further developments in the circle-valued Morse theory. The exposition here is more rapid and we do not aim at a systematic treatment of the subject. I have chosen several topics which are close to my recent research: the Witten framework for the Morse theory, the theory of fibrations of manifolds over a circle and the circle-valued Morse theory for knots and links.

Brief historical comments can be found in the concluding sections of Parts 2, 3 and 4, and some more remarks are scattered through the text. However I did not aim to present a complete historical overview of the subject, and I apologize for possible oversights.

The book is accessible for 1st year graduate students specializing in geometry and topology. Knowledge of the first chapters of the textbooks of M. Hirsch [**61**] and A. Dold [**29**] is sufficient for understanding most of the book. When we need more material, a brief introduction to the corresponding theory is included. This is the case for the Hadamard-Perron theorem (Chapter 4) and the theory of Whitehead torsion (Chapter 13).

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Moscow, August 2006 Andrei Pajitnov

Introduction

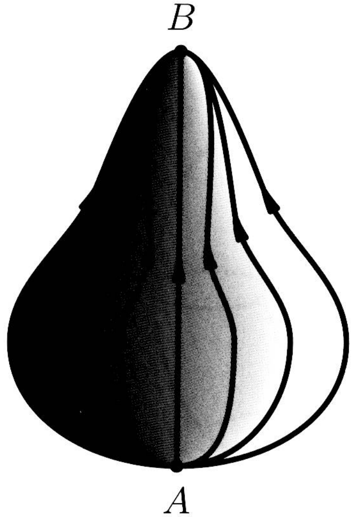
A C^∞ function $f : M \to \mathbf{R}$ on a closed manifold M must have at least two critical points, namely maximum and minimum. This lower bound for the number of critical points is far from exact: the existence of a function on M with precisely two critical points implies a strong restriction on the topology of M. Indeed, let v be the gradient of f with respect to some Riemannian metric, so that

$$\langle v(x), h \rangle = f'(x)(h)$$

for every $x \in M$ and every $h \in T_x M$ (here $\langle \ , \ \rangle$ denotes the scalar product induced by the Riemannian metric). Assuming that f has only two critical points: the minimum A and the maximum B, the vector field v has only two equilibrium points: A and B, and it is not difficult to see that every non-constant integral curve γ of v has the following property:

$$\lim_{t \to \infty} \gamma(t) = B, \qquad \lim_{t \to -\infty} \gamma(t) = A.$$

Therefore the one-point subset $\{B\}$ is a deformation retract of the subset $M \setminus \{A\}$. The deformation retraction is shown in the next figure:

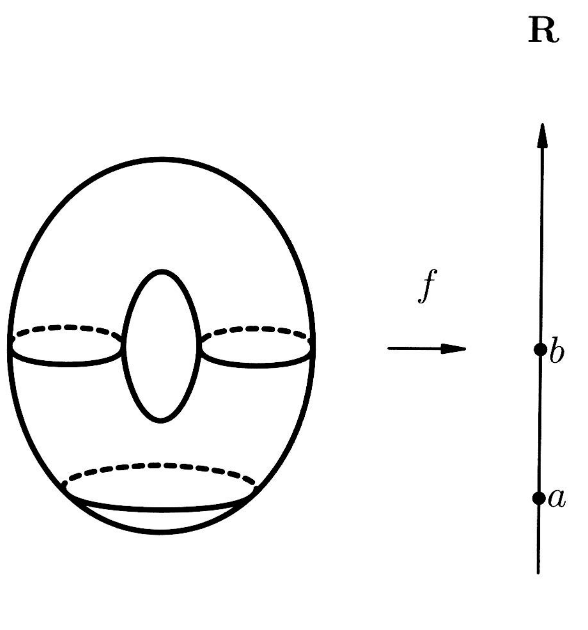


$M \setminus \{A\}$ is deformed onto B along the flow lines of v. In particular $M \setminus \{A\}$ is contractible, and it is not difficult to deduce that M is a homological sphere.

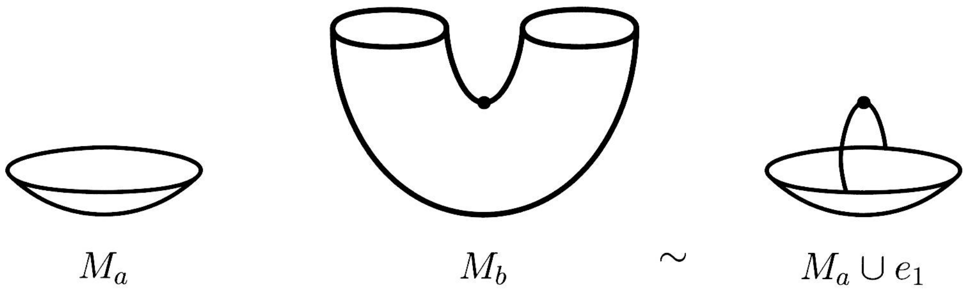
This example suggests that the homology of M can provide efficient lower bounds for the number of critical points of a C^∞ function on a manifold. Such estimates were established by M. Morse in his seminal paper [**98**]. Here is an outline of his discovery. Recall that a critical point p of a function f is called *non-degenerate* if the matrix of the second order partial derivatives of f at p is non-degenerate. The number of the negative eigenvalues of this matrix is called the *index of* p. We shall consider only C^∞ functions whose critical points are all non-degenerate (*Morse functions*). Let $f : M \to \mathbf{R}$ be such a function. Put

$$M_a = \{x \in M \mid f(x) \leqslant a\}.$$

M. Morse shows that if an interval $[a, b]$ contains no critical values of f, then M_a has the same homotopy type as M_b. If $f^{-1}([a, b])$ contains one critical point of f of index k, then M_b has the homotopy type of M_a with one k-cell attached. The classical example below illustrates this principle. Here M is the 2-dimensional torus $\mathbf{T}^2$ embedded in $\mathbf{R}^3$, and f is the height function.



The homotopy type of M_b is clearly the homotopy type of M_a with a one-dimensional cell e_1 attached:



Returning to the general case, it is not difficult to deduce that the manifold M has the homotopy type of a CW complex with the number of k-cells equal to the number $m_k(f)$ of critical points of f of index k. This leads to the *Morse inequalities*:

$$m_k(f) \geqslant b_k(M) + q_k(M) + q_{k-1}(M)$$

where $b_k(M)$ is the rank of $H_k(M)$ and $q_k(M)$ is the torsion number of $H_k(M)$, that is, the minimal possible number of generators of the torsion subgroup of $H_k(M)$. (This version of the Morse inequalities is due to E. Pitcher [**125**]; it is slightly different from Morse's original version.) The applications of these results are too numerous to cite here; we will mention only the classical theorem of M. Morse on the infinite number of geodesics joining two points of a sphere S^n (endowed with an arbitrary Riemannian metric) and the computation by R. Bott of the stable homotopy groups of the unitary groups.

The construction described above can be developed further. Intuitively, it is possible not only to obtain the number of cells of a CW complex X homotopy equivalent to M, but also to compute the boundary operators in the corresponding cellular chain complex. In more precise terms, starting with a Morse function $f : M \to \mathbf{R}$ and an f-gradient v, one can construct a chain complex $\mathcal{M}_*$ such that $\mathcal{M}_k$ is the free abelian group generated by critical points of f of index k and the homology of $\mathcal{M}_*$ is isomorphic to $H_*(M)$.

The explicit geometric construction of $\mathcal{M}_*$ is a result of a long development of the Morse theory (especially in the works of R. Thom [**157**], S. Smale [**150**] [**149**], and E. Witten [**163**]). By definition, $\mathcal{M}_k$ is the free abelian group generated by the set $S_k(f)$ of all critical points of f of index k. The boundary operator $\mathcal{M}_k \to \mathcal{M}_{k-1}$ is defined as follows. Let v be the Riemannian gradient for f with respect to a Riemannian metric on M. For two critical points p, q of f with $\operatorname{ind} p = \operatorname{ind} q + 1$, denote by $\Gamma(p, q; v)$

the set of all flow lines of $(-v)$ from p to q. It turns out that under some natural transversality condition on the gradient flow, this set is finite. The gradients satisfying this condition are called *Kupka-Smale gradients*, they form a dense subset in the space of all gradients of f. One can associate a sign $\varepsilon(\gamma) = \pm 1$ to each flow line γ of $(-v)$ joining p with q (we postpone all the details to Chapters 4 and 6). Summing up the signs we obtain the so-called *incidence coefficient* of p and q:

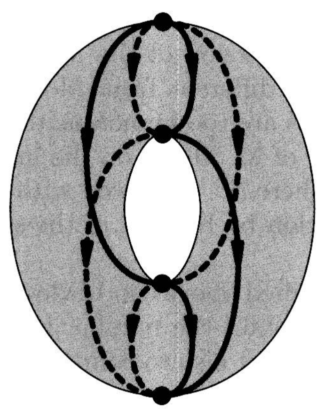
$$n(p,q;v) = \sum_{\gamma \in \Gamma(p,q;\, v)} \varepsilon(\gamma).$$

Now we define the boundary operator $\partial_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ as follows:

$$\partial_k(p) = \sum_{q \in S_{k-1}(f)} n(p,q;v)q.$$

One can prove that $\partial_k \circ \partial_{k+1} = 0$ for every k and the homology of the resulting complex is isomorphic to $H_*(M)$. This chain complex is called *the Morse complex.*

Here is a picture which illustrates the 2-torus case, considered above:



There are four critical points: one of index 0 (the minimum), one of index 2 (the maximum), and two critical points of index 1 (saddle points). There are eight flow lines of $(-v)$ joining the critical points of adjacent indices; they are shown in the figure by curves with arrows. The Morse complex is as follows:

$$0 \longleftarrow \mathbf{Z} \longleftarrow \mathbf{Z}^2 \longleftarrow \mathbf{Z} \longleftarrow 0$$

where all boundary operators are equal to 0.

It is natural to consider also the *circle-valued Morse functions*, that is, C^∞ functions with values in S^1 having only non-degenerate critical points. Identifying the circle with the quotient $\mathbf{R}/\mathbf{Z}$ we can think of circle-valued