HANDBOOK
of
NUMERICAL ANALYSIS

P. G. CIARLET *and* J. L. LIONS • Editors

Volume
IV

Finite Element Methods (Part 2)

Numerical Methods for Solids (Part 2)

NORTH-HOLLAND

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Handbook of Numerical Analysis

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General Preface

During the past decades, giant needs for ever more sophisticated mathematical models and increasingly complex and extensive computer simulations have arisen. In this fashion, two indissociable activities, *mathematical modeling* and *computer simulation*, have gained a major status in all aspects of science, technology, and industry.

In order that these two sciences be established on the safest possible grounds, mathematical rigor is indispensable. For this reason, two companion sciences, *Numerical Analysis* and *Scientific Software*, have emerged as essential steps for validating the mathematical models and the computer simulations that are based on them.

Numerical Analysis is here understood as the part of *Mathematics* that describes and analyzes all the numerical schemes that are used on computers; its objective consists in obtaining a clear, precise, and faithful, representation of all the "information" contained in a mathematical model; as such, it is the natural extension of more classical tools, such as analytic solutions, special transforms, functional analysis, as well as stability and asymptotic analysis.

The various volumes comprising the *Handbook of Numerical Analysis* will thoroughly cover all the major aspects of Numerical Analysis, by presenting accessible and in-depth surveys, which include the most recent trends.

More precisely, the Handbook will cover the *basic methods of Numerical Analysis*, gathered under the following general headings:

- Solution of Equations in $\mathbb{R}^n$,
- Finite Difference Methods,
- Finite Element Methods,
- Techniques of Scientific Computing,
- Optimization Theory and Systems Science.

It will also cover the *numerical solution of actual problems of contemporary interest in Applied Mathematics*, gathered under the following general headings:

- Numerical Methods for Fluids,
- Numerical Methods for Solids,
- Specific Applications.

"Specific Applications" include: Meteorology, Seismology, Petroleum Mechanics, Celestial Mechanics, etc.

Each heading is covered by several *articles*, each of which being devoted to a specialized, but to some extent "independent", topic. Each article contains a thorough description and a mathematical analysis of the various methods in actual use, whose practical performances may be illustrated by significant numerical examples.

Since the Handbook is basically expository in nature, only the most basic results are usually proved in detail, while less important, or technical, results may be only stated or commented upon (in which case specific references for their proofs are systematically provided). In the same spirit, only a "selective" bibliography is appended whenever the roughest counts indicate that the reference list of an article should comprise several thousand items if it were to be exhaustive.

Volumes are numbered by capital Roman numerals (as Vol. I, Vol. II, etc.), according to their *chronological appearance*.

Since all the articles pertaining to a given *heading* may not be simultaneously available at a given time, a given heading usually appears in more than one volume; for instance, if articles devoted to the heading "Solution of Equations in $\mathbb{R}^n$" appear in Volumes I and III, these volumes will include "Solution of Equations in $\mathbb{R}^n$ (Part 1)" and "Solution of Equations in $\mathbb{R}^n$ (Part 2)" in their respective titles. Naturally, all the headings dealt with within a given volume appear in its title; for instance, the complete title of Volume I is "Finite Difference Methods (Part 1)—Solution of Equations in $\mathbb{R}^n$ (Part 1)".

Each article is subdivided into *sections*, which are numbered consecutively throughout the article by *Arabic numerals*, as Section 1, Section 2, . . . , Section 14, etc. Within a given section, *formulas*, *theorems*, *remarks*, *and figures*, have their own independent numberings; for instance, with Section 14, formulas are numbered consecutively as (14.1), (14.2), etc., theorems are numbered consecutively as Theorem 14.1, Theorem 14.2, etc. For the sake of clarity, the article is also subdivided into *chapters*, numbered consecutively throughout the article by *capital Roman numerals*; for instance, Chapter I comprises Sections 1 to 9, Chapter II comprises Sections 10 to 16, etc.

P.G. CIARLET
J.L. LIONS
May 1989

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Finite Element Methods
(Part 2)

Origins, Milestones and Directions of the Finite Element Method—A Personal View

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HANDBOOK OF NUMERICAL ANALYSIS, VOL. IV
Finite Element Methods (Part 2)—Numerical Methods for Solids (Part 2)
Edited by P.G. Ciarlet and J.L. Lions

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CHAPTER I

Origins, Milestones and Directions of the Finite Element Method—A Personal View

1. Introduction

It is now over thirty years since I became involved in "the finite element method" which during most of that period dominated my research activity. The invitation to write this article provides me with a most welcome opportunity to record the story of its origins and of its subsequent development, highlighting the important milestones and directions. Clearly the latter parts are very much a personal view and hence selective. Apologies are given in advance for omission of those who perhaps may view other steps as more important. A full and thorough mathematical analysis of the finite element method as well as many more theoretically oriented references will be found in Volume II of this Handbook.

Since my early introduction to the possibilities offered by numerical approximation by Sir Richard SOUTHWELL [1940, 1946, 1956] viz. his relaxation methods and Allen [1955], my objective has been always that of providing solutions for otherwise intractible problems of interest to applied science and engineering. This objective indeed was shared by others with similar background and led to the development of the finite element method in the late fifties and sixties.

This method was only made possible by the advent of the electronic, digital computer which at the time was making its entry into the field of large arithmetic processing. Indeed the rapid rise and widespread recognition of the methodology of Finite Elements is clearly linked with the development of the computer. This of course led to a rapid development of the method which today, through various commercial and research codes, provides the key for rational design of structures, study of aeronautical fluid dynamics and electromagnetic devices needed by physics.

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