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Mathematical Models for the Analysis and Optimization of Elastoplastic Structures

A. A. CYRAS

ANALYSIS AND OPTIMIZATION OF ELASTOPLASTIC SYSTEMS

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Preface

The application of computer techniques in design practice requires an accurate mathematical formalization of the problems of designing structures. This is particularly important in the creation of automated design systems. The rigorous mathematical formalization of problems in the mechanics of a solid deformable body, together with the use of general methods of discretization, is the foundation of a successful application of any design theory to the practice of designing real structures. At the present time, design methods that take account of the plastic properties of materials do not have these factors to an adequate degree, and this is a serious drawback to their introduction. Since these methods enable us to create more economic structures, it is especially important to overcome this gap in science.

The aim of this book is to combine methods of constructing mathematical models for the design of different elastoplastic systems, both in respect of the design aim and in respect of different actions of external loads. The resulting mathematical models are reduced to dual pairs of mathematical programming problems, and as a result, have one and the same form for different methods of discretization. These models can be harnessed to algorithms for the automated design of buildings. In this book we devote the main attention to the rigorous mathematical formalization of problems, rather than to the physical side, which has been discussed fairly extensively by various authors.

The bibliography, which does not pretend to be an exhaustive coverage of this question, mentions only those publications that have been used directly in the study.

The author thanks his former postgraduate student R. P. Karkauskas for his participation in the development of certain problems in Chapter 5.

A. A. Čyras

Introduction

An important problem of our national economy is to design structures taking account of the plastic properties of materials. The inclusion of plastic properties enables one to foresee more correctly the behaviour of a structure at different stages of a loading, and to produce a more rational design. During the last decade, a significant step in the design of elastoplastic structures has been the mathematical formalization of the design problem, which has opened up the possibility of using computer techniques with the aim of realizing practical design problems.

The calculation of plastic properties of materials in designing structures can be carried out in various ways. In this book we consider three basic types of problems.

The most common problem is to determine the carrying capacity of a structure, that is, the magnitude of the load under which deformations of the structure grow indefinitely without an increase in the load. Such problems are usually called limit equilibrium problems; all the parameters of the structure are assumed to be known, and the only unknown is the parameter of the load producing plastic failure. The limit equilibrium problem, which is classical in applied plasticity theory, apparently arose in connection with the need to compare the results of theoretical and experimental evaluations of failure loads. This problem is least of all suitable for direct design practice since it is essentially a purely verification problem, that is, we require to know all the parameters of a structure beforehand.

The second type of problem is related to the optimization problems of the mechanics of a deformable solid body. Here we determine as well as the strain-deformed state, the parameters of the structure or of the load that correspond to a specific objective function. In design practice the most widespread and real problem is to determine the optimal distribution of the limit forces which characterize the carrying capacity of the separate cross-sections of a given form of structure. The present investigation is devoted to this problem.

The third type of problem is connected with the analysis of a structure undergoing plastic deformations but not reaching complete plastic failure. The determination of the strain-deformed state at any stage of the loading, or after the system is unloaded, is very important both for design and for operation. It should be noted that in a given problem the structure and load parameters must be known completely.

The behaviour of a structure depends on the form of the action of a load. We discuss three basic forms of quasistatic loading: monotonically increasing (simple), cyclic, and movable; they are defined fully in the relevant sections of the book. Here we only mention that the actual strain-deformed state of a structure is determined by using the appropriate extremum energy principles of the mechanics of a perfectly plastic body. The mathematical formalization of these principles for the separate forms of the loading leads to mathematical models of the problems, which can be formed for continuum and discrete systems. In the first case the strain-deformed state is described in function spaces, and in the second, in finite-dimensional spaces. The mathematical models in function spaces are more general and serve as a basis for the derivation of mathematical models for the design of specific structures. Essentially, models in function spaces are used mainly for a qualitative analysis of general problems, that is, to determine the general properties of their solutions. For the practical design of concrete systems it is more suitable to use discrete mathematical models; these are formed under specific assumptions which on the one hand influence the accuracy of the results, and on the other clear the way for solving problems by numerical methods and computer techniques. Therefore, in this book, we do not present the basic mathematical models for continuum systems in function spaces, but restrict ourselves to a discrete finite-dimensional system characterized by generalized forces, deformations, displacements, and a load in finite-dimensional spaces. We use a vector-matrix notation for these variables and also for specific relationships; this notation is not only convenient for computer application, but is intuitive for the formation and the analysis of mathematical models of problems that are especially related to problems of mathematical programming, to which all the problems in question reduce. We mention that the reduced discrete mathematical models are suitable for any discretization method (finite elements, finite differences, etc.).

In the investigations we use repeatedly the deductions of the duality theory of mathematical programming, in which a given extremum problem is compared with the closely related dual problem. A joint consideration of both dual problems is fruitful for a qualitative investigation of extremum problems, and in estimating the accuracy of solutions. The situation is that the solution of problems in force variables and the corresponding description of the unknown quantities in any discretization method, gives a lower bound for the limit load parameter, the theoretical weight, etc., while the solution of problems in the variables of the deformed state gives the upper bound. For example, in using the finite element method, one can judge the advisability of applying one or another form of elements, and also the form of the coordinate functions. Finally, it should be noted that the matrix formulation of problems enables us to apply our results to the design of complicated structures, and for the production of standard programs for computer design.

CHAPTER 1

A discrete description of elastoplastic structures

1.1 MAIN ASSUMPTIONS AND NOTATION

In constructing mathematical models we make the following assumptions:

1. The application of all forms of loading refers to a quasistatic type, that is, dynamical effects are not taken into account in the mathematical models. The calculation of inertia forces does not change the structure of the models, and is carried out in the same way as in problems in elasticity theory.

2. The material of a structure is perfectly plastic and isotropic. As is well known, perfect plasticity is the first approximation to the real behaviour of a system at the elastic limit, and methods based on it are used, as a rule,when the exhaustion of the carrying capacity is considered to be the limiting state of a structure. This idealization of the material may seem fairly crude, but experiments show that even for materials such as reinforced concrete, the use of the limit equilibrium method in designing a system, not only corresponds to the concept of a limiting state, but also gives a definite economy in determining the dimensions of systems in comparison with the method of elastic design.

3. Deformations at plastic failure are small, and so the equations of equilibrium are formed for undeformed structures, that is, we solve a geometrically linear problem.

These assumptions are applied in the construction of mathematical models of a continuum discrete structure. When considering specific systems (frames, plates, shells), it is usual to take into account the additional assumptions determined by the technical theory of the design. These are well known, and have no influence on the structure of the mathematical models.

We define certain basic concepts concerning a discrete model of a structure.

Suppose that an elastoplastic body is partitioned into a finite set of computable elements with indices $k = 1, 2, \ldots, v$; the term 'computable element' is used in a wide sense: for the finite element technique it is a finite element; for the method of finite differences it is a nodal point of a grid; for frameworks it is simply a rod. Thus, a real system is replaced by a discrete model. In choosing the generalized forces and kinematic variables that characterize the strain-deformed state of a discrete model, it is necessary simultaneously to take account of static and kinematic conditions.

Thus, suppose that the strained state at any point $\mathbf{x} \equiv (x_1, x_2, x_3)$ of a computable element is characterized by an n_k-dimensional force vector

$$\mathbf{S}^{kx} \equiv (S_j^{kx}) \equiv (S_1^{kx},\ S_2^{kx},\ \ldots,\ S_{n_k}^{kx})^{\mathrm{T}}. \tag{1.1}$$

The dual formulation of the problem requires the deformation at the same point $\mathbf{x}$ to be defined by an n_k-dimensional vector

$$\mathbf{q}^{kx} \equiv (q_j^{kx}) \equiv (q_1^{kx},\ q_2^{kx},\ \ldots,\ q_{n_k}^{kx})^{\mathrm{T}}. \tag{1.2}$$

This is connected with the fact that the scalar product of the above two vectors must be equal to the dissipation of energy at the same point of a computable element:

$$\mathcal{D}^{kx} = (\mathbf{q}^{kx})^{\mathrm{T}}\, \mathbf{S}^{kx}. \tag{1.3}$$

At this stage we specify the number of points being considered in a computable element. Suppose that the number of them for the whole structure is n, then the vectors

$$\mathbf{S} \equiv (S_j) \equiv (\mathbf{S}^1,\ \mathbf{S}^2,\ \ldots,\ \mathbf{S}^v)^{\mathrm{T}}, \tag{1.4}$$

$$\mathbf{q} \equiv (q_j) \equiv (\mathbf{q}^1,\ \mathbf{q}^2,\ \ldots,\ \mathbf{q}^v)^{\mathrm{T}} \tag{1.5}$$

uniquely define the strain deformed state of the discrete structure, whose dissipation energy is the scalar product of these two vectors

$$\mathcal{D} = \mathbf{q}^{\mathrm{T}}\, \mathbf{S}. \tag{1.6}$$

Another pair of dual variables consists of the loading and the displacement.

Suppose that the displacements at the point in question of a computable element are defined by the m_k-dimensional vector

$$\mathbf{u}^{kx} \equiv (u_i^{kx}) \equiv (u_1^{kx},\ u_2^{kx},\ \ldots,\ u_{m_k}^{kx})^{\mathrm{T}}. \tag{1.7}$$

Then the loading at this point must be defined by a vector of the same dimension

$$\mathbf{F}^{kx} \equiv (F_i^{kx}) \equiv (F_1^{kx},\ F_2^{kx},\ \ldots,\ F_{m_k}^{kx})^{\mathrm{T}}. \tag{1.8}$$

For all the computable elements of the structure we have the two m-dimensional vectors

$$\mathbf{u} \equiv (u_i) \equiv (\mathbf{u}^1,\ \mathbf{u}^2\ \ \ldots,\ \mathbf{u}^v)^{\mathrm{T}},$$

$$\mathbf{F} \equiv (F_i) \equiv (\mathbf{F}^1,\ \mathbf{F}^2,\ \ldots,\ \mathbf{F}^v)^{\mathrm{T}}. \tag{1.9}$$

The work done by the external loading for the whole discrete structure is given by the scalar product of these two vectors

$$W = \mathbf{u}^{\mathrm{T}}\,\mathbf{F}. \tag{1.10}$$

Thus, we have defined the following vectors for a discrete structure: a force vector **S**, a deformation vector **q**, an external loading **F**, and a displacement vector **u**. These symbols refer to actual forces, deformations, and displacements, that is, those arising in a calculation of the elastoplastic work of a structure at a specific stage of the loading, If we obtain their values by taking into account only the elastic state of a structure, then we attach a subscript '*e*' to the corresponding vectors: ($\mathbf{S}_e$, $\mathbf{q}_e$, $\mathbf{u}_e$). If we take into account the residual forces, deformations, or displacements, then the appropriate letter carries a subscript '*r*': ($\mathbf{S}_r$, $\mathbf{q}_r$, $\mathbf{u}_r$).

As is well known, in plasticity theory, the concepts of actual deformations and displacements are connected with their rates of change, that is, with their increments in unit time; we denote these by a dot over the appropriate letter ($\dot{\mathbf{q}}$, $\dot{\mathbf{u}}$).

1.2 MAIN RELATIONSHIPS

We turn to the formation of the main relationships entering into the mathematical modelling of problems.

If the pairs of dual vectors **S** and **q** or **F** and **u** are chosen as indicated above, then the equilibrium equations of a discrete structure have the form

$$[A]\,\mathbf{S} = \mathbf{F}, \tag{1.11}$$

where $[A]$ is the algebraic operator of the equilibrium equations for the whole discrete structure, and can be derived by using a suitable discretization method. It should be noted that the operator $[A]$ is obtained on the basis of the differential operator of the equilibrium equations for an elementary volume of the corresponding continuum structure; for a framework it is formed directly by using the algebraic equilibrium equations.

The geometric equations, which define the connection between displacements and deformations, can be obtained purely formally since the operators of the equilibrium equations and of kinematic compatibility are adjoints. Thus we have

$$[A]^{\mathrm{T}}\,\mathbf{u} = \mathbf{q}, \tag{1.12}$$

where $[A]^{\mathrm{T}}$ is the transpose of the operator of the equations of equilibrium. The dimensions of the vectors **u** and **F** give the number of possible displacements of the whole structure, and consequently, to each equilibrium equation must correspond a kinematic variable u_j, and to each geometric equation a dynamic variable S_j.

As is known, the equilibrium equations and geometric equations do not depend on the properties of the material, and have one and the same form for any state, including for residual forces, deformations, and displacements. The only difference is that by the physical concept of residual forces, they are self-balanced, and the right-hand side of the equilibrium equations is the zero vector, that is,

$$[A]\,\mathbf{S}_r = \mathbf{0}. \tag{1.13}$$

The geometric equations for residual deformations and displacements are unchanged.

We define the physical relationships for a discrete model.

The behaviour of the material of a body in a complicated strained state is described by a yield function whose argument consists of the forces. Suppose that for a point $\mathbf{x}$ of a computable element k this function is a scalar function and convex from below. Then, for any point of a computable element, the yield conditions have the form

$$f^{kx}(\mathbf{S}^{kx}) \leqslant S_0^k, \tag{1.14}$$

where $\mathbf{S}_0^k$, which is the limit force on the kth element, is a function of the yield limit σ_S of the material and of a characteristic dimension of the element itself, for example, the thickness of a plate or a shell. Conditions of the type (1.14) can be written out for every point being considered in a discrete structure, or for every computable element. Without loss of generality, we may assume that the yield conditions for the whole discrete structure are described by a vector function

$$\mathbf{f} \equiv (f_l), \qquad l = 1, 2, \ldots, t.$$

Then the yield conditions for a structure have the form

$$\mathbf{f}(\mathbf{S}) \leqslant \mathbf{S}_0, \tag{1.15}$$

where $\mathbf{S}_0$ is the vector of length t of the limit forces. The multiplying proportionality vector has the same dimension:

$$\boldsymbol{\lambda} \equiv (\lambda_l).$$

Thus, the dissipation of energy has the form (with accuracy up to a constant multiplier):

$$\mathcal{D} = \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{S}_0. \tag{1.16}$$

By applying the rule of plastic flow, we obtain a connection between the force vector and the velocity vector of the deformation

$$\dot{\mathbf{q}} = \left[\frac{\partial \mathbf{f}(\mathbf{S})}{\partial \mathbf{S}}\right]^{\mathrm{T}} \boldsymbol{\lambda} = [\tilde{f}(\mathbf{S})]\,\boldsymbol{\lambda}, \tag{1.17}$$

where $[\tilde{f}(\mathbf{S})]$ is the $n \times t$-matrix of the gradients of the yield vector-function, and is given by

$$[\tilde{f}(\mathbf{S})] \equiv [\tilde{f}_{jl}] \equiv \left[\frac{\partial \mathrm{f}_l}{\partial S_j}\right], \qquad j = 1,\ 2,\ \ldots,\ n;\ l = 1,\ 2,\ \ldots,\ t. \qquad (1.18)$$

The expressions we have obtained enable us to write down the following relationships of flow theory, valid up to a constant multiplier:

$$\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{S} = \boldsymbol{\lambda}^{\mathrm{T}}\,[\tilde{f}(\mathbf{S})]^{\mathrm{T}}\,\mathbf{S} = \boldsymbol{\lambda}^{\mathrm{T}}\,\mathbf{f}\,(\mathbf{S}) = \boldsymbol{\lambda}^{\mathrm{T}}\,\mathbf{S}_0. \qquad (1.19)$$

Since $\mathbf{S}_0$ and the multiplier $\boldsymbol{\lambda}$, which performs the role of a scale, are positive, the expressions (1.16) and (1.19) show that the rate of dissipation of energy is essentially positive.

Finally, it should be noted that in contrast to problems in elasticity theory, in the solutions of plasticity problems there can be discontinuities of the forces and rates of deformation between computable elements. In this case, we must add to the expressions for the rate of dissipation of energy and the magnitude of the loading, the dissipation and magnitude at places of discontinuity.

SUMMARY OF CHAPTER 1

We consider a discrete elastoplastic structure. The stress-strain field of the structure is defined by the force vector $\mathbf{S}$, the vector of generlaized strains $\mathbf{q}$, and the vector of generalized displacements $\mathbf{u}$. The structure is subjected to an external loading defined by the vector $\mathbf{F}$. The main relationships are presented in the form of the equilibrium equations (1.11), the geometric equations (1.12), the yield conditions (1.15), and the physical law (1.17). A continuum system may be reduced to these relationships by any conventional discretization method, for example, by the finite element technique, by the method of finite differences, etc.

CHAPTER 2

Mathematical models of limit equilibrium problems

In a given problem we shall assume that all the parameters of a structure are completely known, that is, we know its configuration, dimensions, and also the limit force vector $\mathbf{S}_0$. We need to determine the limit load parameter and the strain-deformed state in the plastic failure phase of the structure. Mathematical models of this problem depend mainly on the form of the loading which determines the failure; we consider three forms of loading: monotonically increasing, cyclic, and movable.

2.1 MONOTONICALLY INCREASING LOADING

A monotonically increasing loading is a system of forces, each of which, being proportional to a parameter, increases from zero to a specific value. In the literature this form of loading is also called *simple* or *proportional* loading. Plastic failure of a structure under a monotonically increasing loading results from an accumulation of plastic deformations that leads to their unconstrained growth under a constant load. This form of exhaustion of the carrying capacity is usually called *simple plastic failure*, and the load corresponding to this phase of the work of the structure is known as the *limit load*. Since we assume that the distribution of the load is known, the problem consists in determining the limit load parameter and the strain-deformed state at the instant the carrying capacity is exhausted. At this instant the strained state of the structure is characterized by the forces, and the deformed state by the deformation velocities and displacement velocities. The latter can be explained as follows. Suppose that the plastic failure stage of a structure has been reached, and that under a small increase in the load its deformations and displacements grow unrestrictedly. It is impossible to judge their absolute value at a given instant, but their characteristics can be taken to be increments in plastic deformation and displacement in unit time. Thus, in solving the limit equilibrium problem for the case of a monotonically increasing loading, the required quantities are the forces and velocities of deformation and displacement together with the limit load parameter. We find these