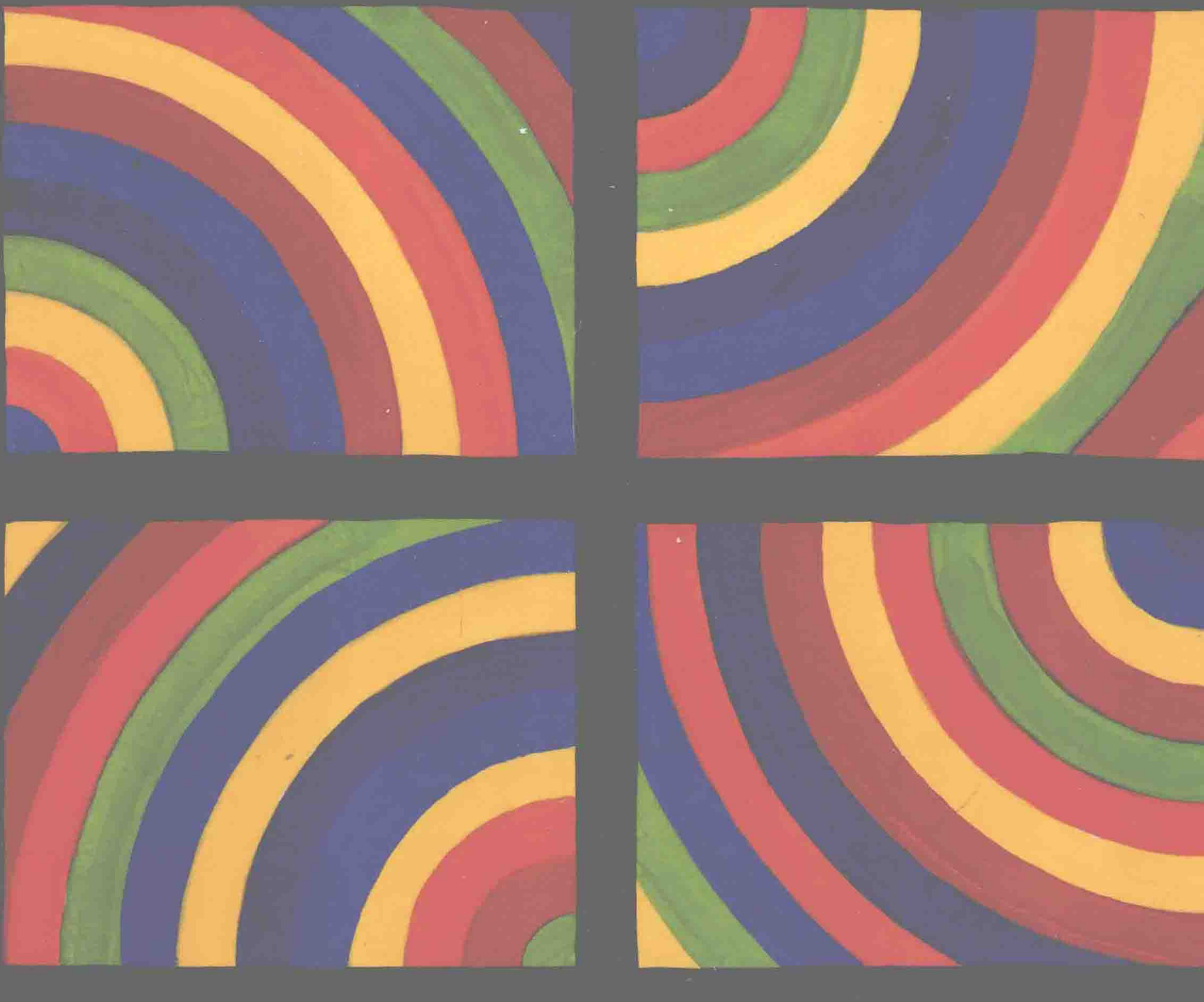
INTRODUCTORY LINEAR ALGEBRA

With Applications

SEVENTH EDITION



BERNARD KOLMAN ▪ DAVID R. HILL

SEVENTH EDITION

INTRODUCTORY LINEAR ALGEBRA

WITH APPLICATIONS

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INTRODUCTORY LINEAR ALGEBRA

WITH APPLICATIONS

To the memory of Lillie
and to Lisa and Stephen

B. K.

To Suzanne

D. R. H.

PREFACE

Material Covered

This book presents an introduction to linear algebra and to some of its significant applications. It is designed for a course at the freshman or sophomore level. There is more than enough material for a semester or quarter course. By omitting certain sections, it is possible in a one-semester or quarter course to cover the essentials of linear algebra (including eigenvalues and eigenvectors), to show how the computer is used, and to explore some applications of linear algebra. The level and pace of the course can be readily changed by varying the amount of time spent on the theoretical material and on the applications. Calculus is not a prerequisite; examples and exercises using very basic calculus are included and these are labeled "Calculus Required."

The emphasis is on the computational and geometrical aspects of the subject, keeping the abstraction down to a minimum. Thus we sometimes omit proofs of difficult or less-rewarding theorems, while amply illustrating them with examples. The proofs that are included are presented at a level appropriate for the student. We have also devoted our attention to the essential areas of linear algebra; the book does not attempt to cover the subject exhaustively.

What Is New in the Seventh Edition

We have been very pleased by the widespread acceptance of the first six editions of this book. By now, the reform movement in linear algebra is in full swing and has resulted in a number of techniques for improving the teaching of linear algebra. The **Linear Algebra Curriculum Study Group** and others have made a number of important recommendations for doing this. In preparing the present edition, we have considered these recommendations as well as suggestions from faculty and students. Although many changes have been made in this edition, our objective has remained the same as in the earlier editions:

> **to develop a textbook that will help the instructor to teach and the student to learn the basic ideas of linear algebra and to see some of its applications.**

To achieve this objective, the following features have been developed in this edition:

- New Sections have been added as follows:
  - Section 2.5, *Introduction to Wavelets*, shows in a simplified manner how the technique of wavelets is used to efficiently transmit large amounts of data.
  - Section 9.3, *Dynamical Systems*, presents an introduction to the qualitative behavior of differential equations.
  - Section 10.4, *Introduction to Fractals*, shows how certain nonlinear transformations have been successfully applied in such diverse areas as special effects in film and television, compression of digital images, meteorology, ecology, biology, and astronomy.

These three sections provide applications of linear algebra to new and exciting areas of applied mathematics.

- Old Section 6.1, *Eigenvalues and Eigenvectors*, has been split into two sections to improve pedagogy.
- More geometric material has been added.
- New exercises at all levels have been added. Some of these are more open-ended, allowing for exploration and discovery, as well as writing.
- More illustrations have been added.

Exercises

The exercises in this book are grouped into three classes. The first class, *Exercises*, contains routine exercises. The second class, *Theoretical Exercises*, includes exercises that fill in gaps in some of the proofs and amplify material in the text. Some of these call for a verbal solution. In this technological age, it is especially important to be able to write with care and precision; therefore, exercises of this type should help to sharpen such skills. These exercises can also be used to raise the level of the course and to challenge the more capable and interested student. The third class consists of exercises developed by David R. Hill and are labeled by the prefix ML (for MATLAB). These exercises are designed to be solved by an appropriate computer software package.

Answers to all odd-numbered numerical and ML exercises appear in the back of the book. At the end of Chapter 10, there is a cumulative review of the introductory linear algebra material presented thus far, consisting of 75 true-false questions (with answers in the back of the book). The **Instructors Solutions Manual**, containing answers to all even-numbered exercises and solutions to all theoretical exercises is available (to instructors only) at no cost from the publisher.

Presentation

We have learned from experience that at the sophomore level, abstract ideas must be introduced quite gradually and must be supported by firm foundations. Thus we begin the study of linear algebra with the treatment of matrices as mere arrays of numbers that arise naturally in the solution of systems of linear equations—a problem already familiar to the student. Much attention has been devoted from one edition to the next to refine and improve the pedagogical aspects of the exposition. The abstract ideas are carefully balanced by the considerable emphasis on the geometrical and computational foundations of the subject.

Material Covered

Chapter 1 deals with matrices and their properties. Methods of solving systems of linear equations are discussed in this chapter. Chapter 2 (optional) discusses applications of linear equations and matrices to the areas of graph theory, electrical circuits, Markov chains, linear economic models, and wavelets. Section 2.3, *Markov Chains*, is new to this edition. Chapter 3 presents the basic properties of determinants rather quickly. Chapter 4 deals with vectors in R^n. In this chapter we also cover vectors in the plane and give an early introduction to linear transformations. Chapter 5 (optional) provides an opportunity to explore some of the many geometric ideas dealing with vectors in R^2 and R^3 ; we limit our attention to the areas of computer graphics, cross product in R^3, and lines and planes.

In Chapter 6 we come to a more abstract notion, that of a vector space. The abstraction in this chapter is more easily handled after the work done with vectors in R^n. Chapter 7 (optional) presents two applications of real vector spaces: QR-factorization and least squares. Chapter 8, on eigenvalues and eigenvectors, the pinnacle of the course, is now presented in three sections to improve pedagogy. The diagonalization of symmetric matrices is carefully developed.

Chapter 9 (optional) deals with a number of diverse applications of eigenvalues and eigenvectors. These include the Fibonacci sequence, differential equations, dynamical systems, quadratic forms, conic sections, and quadric surfaces. Section 9.3, *Dynamical Systems* is new to this edition. Chapter 10 covers linear transformations and matrices. Section 10.4 (optional), *Introduction to Fractals*, which is new to this edition, deals with an application of a certain nonlinear transformation. Chapter 11 (optional) discusses linear programming, an important application of linear algebra. Section 11.4 presents the basic ideas of the theory of games. Chapter 12, provides a brief introduction to MATLAB (which stands for MATRIX LABORATORY), a very useful software package for linear algebra computation, described below.

Appendix A covers complex numbers and introduces, in a brief but thorough manner, complex numbers and their use in linear algebra. Appendix B presents two more advanced topics in linear algebra: inner product spaces and composite and invertible linear transformations.

Applications

Most of the applications are entirely independent; they can be covered either after completing the entire introductory linear algebra material in the course or they can be taken up as soon as the material required for a particular application has been developed. Brief Previews of most applications are given at appropriate places in the book to indicate how to provide an immediate application of the material just studied. The chart at the end of the Preface giving the prerequisites for each of the applications and the Brief Previews will be helpful in deciding which applications to cover and when to cover them.

Some of the sections, in Chapters 2, 5, 7, 9, and 11 can also be used as independent student projects. Classroom experience with the latter approach has met with favorable student reaction. Thus the instructor can be quite selective both in the choice of material and in the method of study of these applications.

End of Chapter Material

Every chapter contains a summary of *Key Ideas for Review*, a set of Supplementary Exercises (answers to all odd-numbered exercises appear in the back of the book), and a Chapter Test (all answers appear in the back of the book).

MATLAB Software

Although the ML exercises can be solved using a number of software packages, in our judgment MATLAB is the most suitable package for this purpose. MATLAB is a versatile and powerful software package whose cornerstone is its linear algebra capability. MATLAB incorporates professionally-developed quality computer routines for linear algebra computation. The code employed by MATLAB is written in the C language and is upgraded as new versions of MATLAB are released. MATLAB is available from The Math Works, Inc., 24 Prime Park Way, Natick, MA 01760, (508) 653-1415; e-mail: **info@mathworks.com** and is not distributed with this book or the instructional routines developed for solving the ML exercises. The Student Edition of MATLAB also includes a version of Maple, thereby providing a symbolic computational capability.

Chapter 12 of this edition consists of a brief introduction to MATLAB's capabilities for solving linear algebra problems. Although programs can be written within MATLAB to implement many mathematical algorithms, *it should be noted that the reader of this book is not asked to write programs. The user is merely asked to use* MATLAB *(or any other comparable software package) to solve specific numerical problems.* Approximately 18 instructional M-files have been developed to be used with the ML exercises in this book and are available from the following Prentice Hall Website: **www.prenhall.com/kolman**. These M-files are designed to transform many of MATLAB's capabilities into courseware. This is done by providing pedagogy that allows the student to interact with MATLAB, thereby letting the student think through all the steps in the solution of a problem and relegating MATLAB to act as a powerful calculator to relieve the drudgery of a tedious computation. Indeed, this is the ideal role for MATLAB (or any other similar package) in a beginning linear algebra course, for in this course, more than in many others, the tedium of lengthy computations makes it almost impossible to solve a modest-size problem. Thus, by introducing pedagogy and reining in the power of MATLAB, these M-files provide a working partnership between the student and the computer. Moreover, the introduction to a powerful tool such as MATLAB early in the student's college career opens the way for other software support in higher-level courses, especially in science and engineering.

Supplements

Student Solutions Manual (0-13-032852-9). Prepared by Dennis Kletzing, Stetson University, contains solutions to all odd-numbered exercises, both numerical and theoretical. It can be purchased from the publisher.

Instructors Solutions Manual (0-13-032853-7). Contains answers to all even-numbered exercises and solutions to all theoretical exercises—is available (to instructors only) at no cost from the publisher.

Optional combination packages. Provide a MATLAB workbook at a reduced cost when packaged with this book. Any of the following three MATLAB manuals can be wrapped with this text for a small extra charge:

- Hill/Zitarelli, *Linear Algebra Labs with* MATLAB, 2/e (0-13-505439-7).
- Leon/Herman/Faukenberry, *ATLAST Computer Exercises for Linear Algebra* (0-13-270273-8).
- Smith, MATLAB *Project Book for Linear Algebra* (0-13-521337-1).

Prerequisites for Applications

Prerequisites for Applications	
Section 2.1	Section 1.4
Section 2.2	Section 1.5
Section 2.3	Section 1.5
Section 2.4	Section 1.6
Section 2.5	Section 1.6
Section 5.1	Section 4.1
Section 5.2	Section 4.1 and Chapter 3
Section 5.3	Sections 4.1 and 5.2
Section 7.1	Section 6.8
Section 7.2	Sections 1.5, 1.6, 4.2, 6.9
Section 9.1	Section 8.2
Section 9.2	Section 8.2 (Calculus required)
Section 9.3	Section 9.2
Section 9.4	Section 8.3
Section 9.5	Section 9.4
Section 9.6	Section 9.5
Section 10.4	Section 8.2
Sections 11.1–11.3	Section 1.5
Section 11.4	Sections 11.1–11.3

To Users of Previous Editions:

During the 25-year life of the previous six editions of this book, it was primarily used to teach a sophomore level linear algebra course. This course covered the essentials of linear algebra and used any available extra time to study selected applications of the subject. *In this new edition we have not changed the structural foundation for teaching the essential linear algebra material. Thus, this material an be taught in exactly the same manner as before.* The placement of the applications in a more cohesive and pedagogically unified manner together with the newly-added applications and other material should make it easier to teach a richer and more varied course.

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Bernard Kolman
David R. Hill

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CHAPTER 1

LINEAR EQUATIONS AND MATRICES

1.1 LINEAR SYSTEMS

A good many problems in the natural and social sciences as well as in engineering and the physical sciences deal with equations relating two sets of variables. An equation of the type

$$ax = b,$$

expressing the variable b in terms of the variable x and the constant a, is called a **linear equation**. The word *linear* is used here because the graph of the equation above is a straight line. Similarly, the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1}$$

expressing b in terms of the variables $x_1, x_2, \ldots, x_n$ and the known constants $a_1, a_2, \ldots, a_n$, is called a **linear equation**. In many applications we are given b and the constants $a_1, a_2, \ldots, a_n$ and must find numbers $x_1, x_2, \ldots, x_n$, called **unknowns**, satisfying (1).

A **solution** to a linear equation (1) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that (1) is satisfied when $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ are substituted in (1).

Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

This is not the only solution to the given linear equation, since $x_1 = 3$, $x_2 = 1$, and $x_3 = -7$ is another solution.

More generally, a **system of m linear equations in n unknowns x_1, x_2, $\ldots$, x_n**, or simply a **linear system**, is a set of m linear equations each in n unknowns. A linear system can be conveniently denoted by

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{2}$$

The two subscripts i and j are used as follows. The first subscript i indicates that we are dealing with the ith equation, while the second subscript j is associated with the jth variable x_j. Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

In (2) the a_{ij} are known constants. Given values of $b_1, b_2, \ldots, b_m$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **solution** to a linear system (2) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ are substituted in (2).

To find solutions to a linear system, we shall use a technique called the **method of elimination**. That is, we eliminate some of the unknowns by adding a multiple of one equation to another equation. Most readers have had some experience with this technique in high school algebra courses. Most likely, the reader has confined his or her earlier work with this method to linear systems in which $m = n$, that is, linear systems having as many equations as unknowns. In this course we shall broaden our outlook by dealing with systems in which we have $m = n$, $m < n$, and $m > n$. Indeed, there are numerous applications in which $m \neq n$. If we deal with two, three, or four unknowns, we shall often write them as x, y, z, and w. In this section we use the method of elimination as it was studied in high school. In Section 1.5 we shall look at this method in a much more systematic manner.

EXAMPLE 1

The director of a trust fund has \$100,000 to invest. The rules of the trust state that both a certificate of deposit (CD) and a long-term bond must be used. The director's goal is to have the trust yield \$7800 on its investments for the year. The CD chosen returns 5% per annum and the bond 9%. The director determines the amount x to invest in the CD and the amount y to invest in the bond as follows:

Since the total investment is \$100,000, we must have $x + y = 100{,}000$. Since the desired return is \$7800, we obtain the equation $0.05x + 0.09y = 7800$. Thus, we have the linear system

$$\begin{aligned} x + y &= 100{,}000 \\ 0.05x + 0.09y &= 7800. \end{aligned} \tag{3}$$

To eliminate x, we add (-0.05) times the first equation to the second, obtaining

$$0.04y = 2800,$$

an equation having no x term. We have eliminated the unknown x. Then solving for y, we have

$$y = 70{,}000,$$

and substituting into the first equation of (3), we obtain

$$x = 30{,}000.$$

To check that $x = 30{,}000$, $y = 70{,}000$ is a solution to (3), we verify that these values of x and y satisfy *each* of the equations in the given linear system. Thus, the director of the trust should invest \$30,000 in the CD and \$70,000 in the long-term bond. ■

EXAMPLE 2 Consider the linear system

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7. \end{aligned} \tag{4}$$

Again, we decide to eliminate x. We add (-2) times the first equation to the second one, obtaining

$$0 = 21,$$

which makes no sense. This means that (4) has no solution. We might have come to the same conclusion from observing that in (4) the left side of the second equation is twice the left side of the first equation, but the right side of the second equation is not twice the right side of the first equation. ■

EXAMPLE 3 Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned} \tag{5}$$

To eliminate x, we add (-2) times the first equation to the second one and (-3) times the first equation to the third one, obtaining

$$\begin{aligned} -7y - 4z &= 2 \\ -5y - 10z &= -20. \end{aligned} \tag{6}$$

This is a system of two equations in the unknowns y and z. We multiply the second equation of (6) by $\left(-\frac{1}{5}\right)$, obtaining

$$\begin{aligned} -7y - 4z &= 2 \\ y + 2z &= 4, \end{aligned}$$

which we write, by interchanging equations, as

$$\begin{aligned} y + 2z &= 4 \\ -7y - 4z &= 2. \end{aligned} \tag{7}$$

We now eliminate y in (7) by adding 7 times the first equation to the second one, to obtain

$$10z = 30,$$

or

$$z = 3. \tag{8}$$

Substituting this value of z into the first equation of (7), we find $y = -2$. Substituting these values of y and z into the first equation of (5), we find $x = 1$. To check that $x = 1$, $y = -2$, $z = 3$ is a solution to (5), we verify that these values of x, y, and z satisfy *each* of the equations in (5). Thus $x = 1$, $y = -2$, $z = 3$ is a solution to the linear system (5). We might further observe that our elimination procedure has effectively produced the following linear system:

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3, \end{aligned} \tag{9}$$