INTERACTIVE MATHEMATICS

Topic 11
Functions and Graphing

Topic 12
The Exponential and Logarithmic Functions

Topic 13
More Nonlinear Equations and Inequalities

PERSONAL ACADEMIC NOTEBOOK

ACADEMIC SYSTEMS

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Interactive Mathematics

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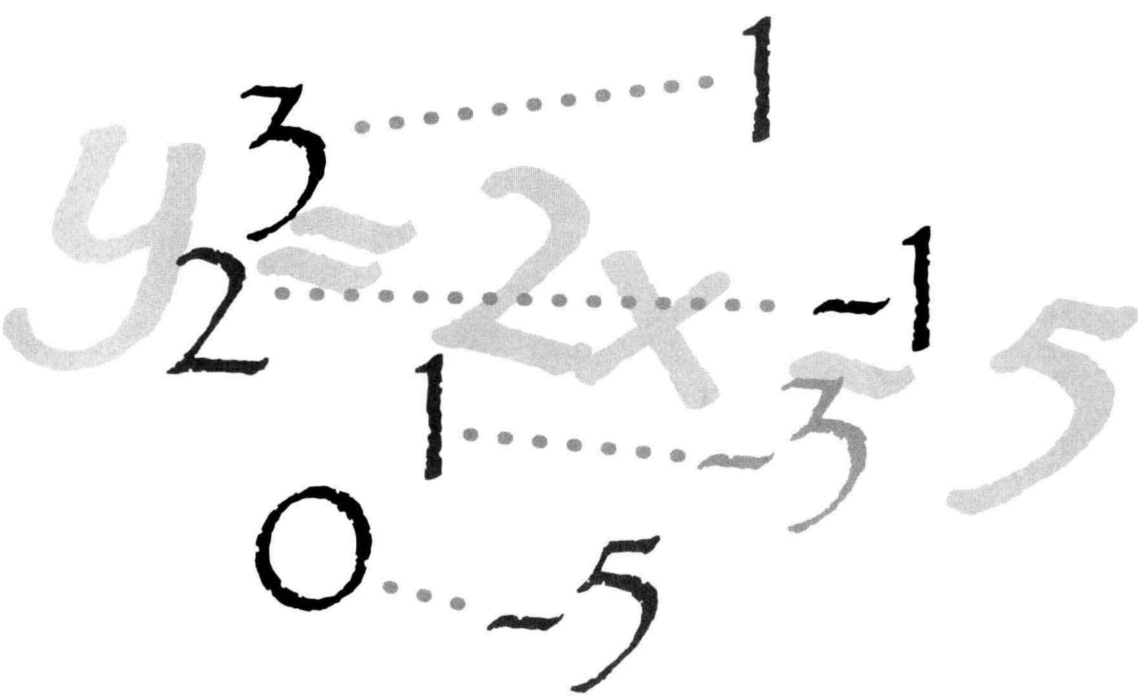
LESSON 11.1 – FUNCTIONS



Here's what you'll learn in this lesson:

Functions and Graphs

a. Definition of a function

b. Function as an ordered pair of numbers

c. Finding function values given a formula

d. Function notation: $y = f(x)$

e. Graphing simple functions

f. Domain and range of a function

g. The vertical line test

Linear Functions

a. Graphs of linear functions

b. Graphs of absolute value functions

Quadratic Functions

a. Graphs of quadratic functions

b. Intercepts of quadratic functions

OVERVIEW

For centuries, artists from many cultures have used patterns to create beautiful mosaics, textiles, and stained glass windows. Many of these patterns can be described mathematically using functions.

In this lesson, you will learn about functions. You will learn how to graph a function, how to identify the domain and the range of a function, and how to use the notation for functions. In addition, you will study three types of functions that you will see again in your study of algebra: linear functions, absolute value functions, and quadratic functions.

EXPLAIN

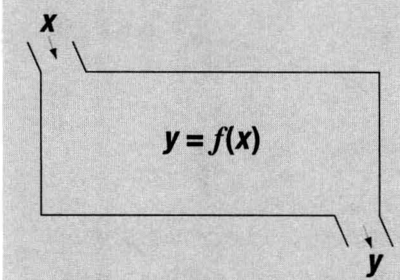
FUNCTIONS AND GRAPHS

Summary

Definition of a Function

Rules such as $y = 3x + 1$ and $y = 2x - 7$ are called functions. These rules assign to each real number x exactly one real number y.

You can think of a function as a machine that assigns to each input number, x, a unique output number, y.



You read "f(x)" as "f of x."

Some examples of functions are:

$y = x + 1$ $\qquad$ $y = x^2$

$y = 4 - 2x$ $\qquad$ $y = \frac{1}{x}$

Sometimes letters such as f, g, or h are used to denote functions. When x is the input number, the output number is written as $f(x)$, $g(x)$, or $h(x)$.

For example, you might write the above functions as:

$f(x) = x + 1$

$g(x) = 4 - 2x$

$h(x) = x^2$

$k(x) = \frac{1}{x}$

To find the value of a function at a given point, x:

1. Substitute the given x-value into the function.
2. Simplify to get the output value.

For example, to find the value of $f(x) = 3x - 7$ at $x = 1$:

1. Substitute 1 for x. $\qquad f(1) = 3(1) - 7$
2. Simplify. $\qquad = 3 - 7$

$\qquad = -4$

As another example, to find the value of $k(x) = \frac{1}{2x}$ at $x = -7$:

1. Substitute -7 for x. $\qquad f(-7) = \frac{1}{2(-7)}$
2. Simplify. $\qquad = \frac{1}{-14}$

$\qquad = -\frac{1}{14}$

*The domain is the x-values. The range is the y-values. Here's a way to help you remember which is which: **d**omain comes before **r**ange in alphabetical order just like **x** comes before **y**.*

Domain and Range of a Function

The **domain** of a function is all of the real numbers, x, for which the function is defined and produces a real number, y. The **range** of a function is all of the possible y-values you can get from the x-values.

The domain and range of a function don't always include all the real numbers.

- When a function includes a square root, the domain does not include any numbers that make the value under the square root negative.

For example, in the function $y = \sqrt{x-1}$ you need to make sure that the $(x - 1)$ under the square root sign is never negative. So the domain includes all nonnegative real numbers greater than or equal to 1.

- When a function includes a fraction, the domain does not include any numbers that make the denominator of the fraction zero.

For example, in the function $y = \frac{1}{x+2}$ you need to make sure that the $(x + 2)$ in the denominator is never zero. So the domain includes all real numbers except –2.

To find the domain and range of a function:

To help you find the domain, start with all real numbers and eliminate those numbers for which the function does not make sense.

1. Find all of the x-values for which the function is defined and produces a real number. This is the domain of the function.
2. Find all possible y-values for all x-values in the domain. This is the range of the function.

For example, to find the domain and range of the function $y = x - 1$:

The domain is all real numbers since $(x - 1)$ is defined for any real number. The range is all real numbers since $(x - 1)$ can equal any real number as x runs through all real numbers.

1. Find all the x-values for which the function is defined.	All real numbers.
2. Find all possible y-values for all x-values in the domain.	All real numbers.

As another example, to find the domain and range of the function $y = \frac{1}{x^2 - 4}$:

The domain can't include 2 or –2 because substituting either of these numbers makes the denominator equal to 0. The range can't include those real numbers which are greater than $-\frac{1}{4}$ and less than or equal to 0. There are no real numbers you can substitute for x that will make $\frac{1}{x^2-4}$ equal to numbers which are greater than $-\frac{1}{4}$ and less than or equal to 0.

1. Find all the x-values for which the function is defined.	Any real number except $x = 2$ or $x = -2$.
2. Find all possible y-values for all x-values in the domain.	All real numbers except those that are greater than $-\frac{1}{4}$ and less than or equal to 0.

Graphing Functions

When you are given a function, you can find ordered pairs of real numbers that satisfy the function and then use these ordered pairs to graph the function.

To find an ordered pair that satisfies a function $y = f(x)$:

1. Pick a value for x.
2. Substitute your chosen value, x, into the function to solve for the other variable, y.
3. Write the ordered pair as (x, y).

For example, to find an ordered pair that satisfies the function $y = 2x - 5$:

1. Pick a value for x. $\quad x = 3$
2. Substitute $x = 3$ into $y = 2x - 5$ and solve for y.

$$y = 2x - 5$$
$$y = 2(3) - 5$$
$$y = 6 - 5$$
$$y = 1$$

3. Write the ordered pair as (x, y). $\quad (3, 1)$

In general, there are infinitely many ordered pairs that satisfy a function. For the function $y = 2x - 5$, for any x you choose you can find a y. Once you have several ordered pairs that satisfy the function, you can plot the points on a grid to graph the function.

To graph a function:

1. Find several ordered pairs that satisfy the function.
2. Plot these points on a grid.
3. Use these points to sketch the graph of the function.

For example, to graph the function $y = 2x - 5$:

1. Find several ordered pairs that satisfy the function.

x	y
3	1
2	–1
1	–3
0	–5

2. Plot these points on a grid.
3. Use these points to sketch the graph of the function. See Figure 11.1.1.

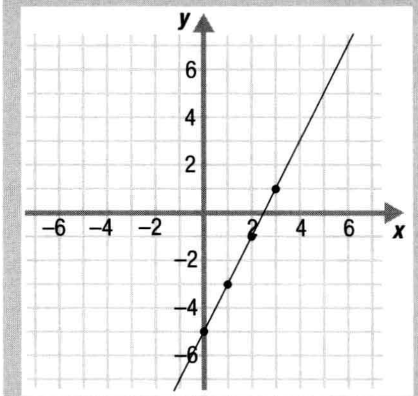


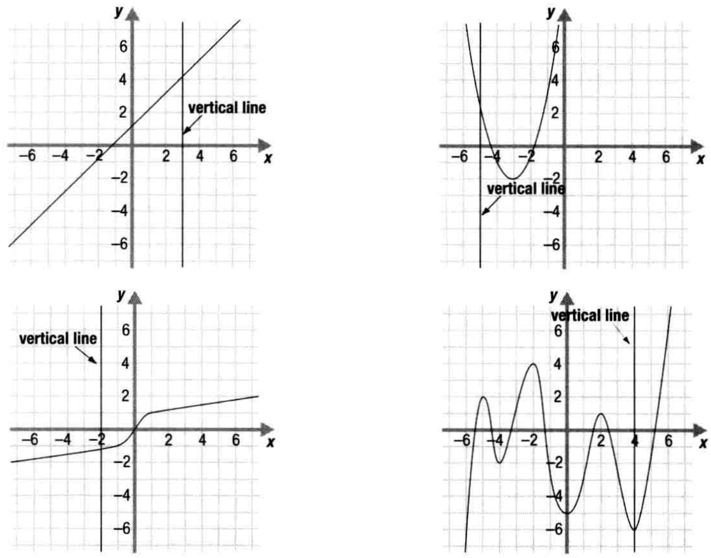
Figure 11.1.1

Vertical Line Test

Remember, a function is a relation that has a unique value, y, for any value of x. Therefore, you can tell if a graph is the graph of a function by using the vertical line test: if you draw a vertical line anywhere through the graph of a function, it will intersect the graph in only one place. Likewise, if a vertical line intersects a graph in more than one place then the graph is not the graph of a function.

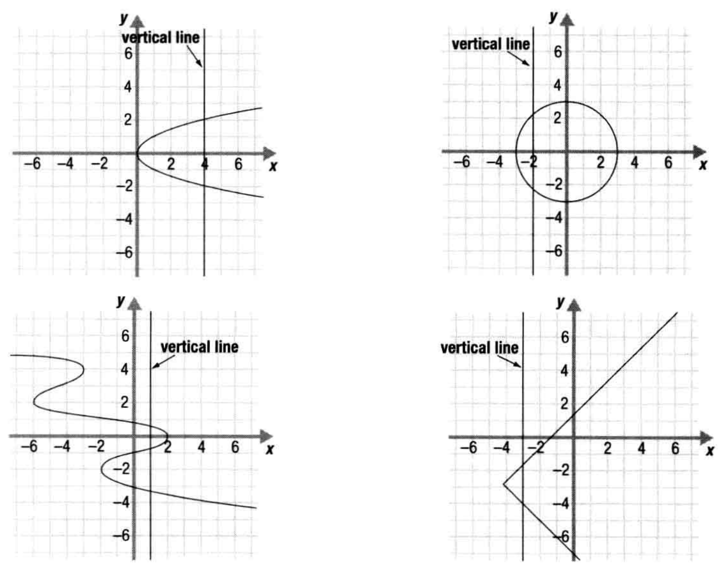
For example, here are the graphs of some functions:

Any vertical line intersects each graph in one and only one place.



Here are some graphs that are not functions:

There is a vertical line that intersects these graphs in more than one place.



Sample Problems

1. For the function $g(x) = 4x - 5$, calculate:

   a. $g(0)$

   b. $g(2)$

   c. $g(-5)$

☑ a. Substitute 0 for x and simplify.

$$g(0) = 4(0) - 5 \\ = 0 - 5 \\ = -5$$

☐ b. Substitute 2 for x and simplify.

$$g(2) = 4(2) - 5 \\ = \_\_\_\_\_\_ \\ = \_\_\_\_\_\_$$

☐ c. Substitute −5 for x and simplify.

$$g(\_\_) = 4(\_\_) - 5 \\ = \_\_\_\_\_\_ \\ = \_\_\_\_\_\_$$

2. Find the domain and range for each of the functions below.

   a. $y = x - 7$

   b. $y = \frac{3}{2x}$

   c. $y = \sqrt{x - 2}$

☑ a. Find all of the x-values for which the function is defined. — domain: all real numbers

Find all possible y-values, given the possible x-values. — range: all real numbers

☐ b. Find all of the x-values for which the function is defined. — domain: ________

Find all possible y-values, given the possible x-values. — range: ________

☐ c. Find all of the x-values for which the function is defined. — domain: ________

Find all possible y-values, given the possible x-values. — range: ________

Answers to Sample Problems

1.
b. 8 − 5
3

c. −5, −5
−20 − 5
−25

2.
b. all real numbers except $x = 0$ ($x \neq 0$)

all real numbers except $y = 0$ ($y \neq 0$)

c. all real numbers greater than or equal to 2 ($x \geq 2$)

all real numbers greater than or equal to 0 ($y \geq 0$)

Answers to Sample Problems

a.

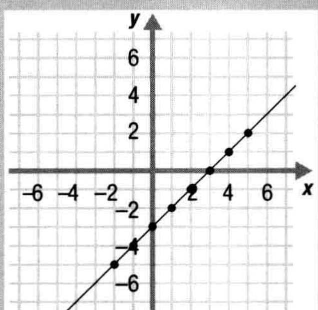
–1

0

1

2

b., c.

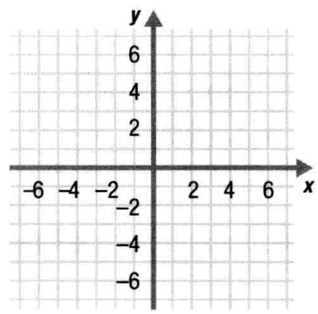


3. Complete the table below for the function $y = x - 3$. Then use the table to graph the function on the grid.

☐ a. Complete the table.

x	$x - 3$
–2	–5
–1	–4
0	–3
1	–2
2	____
3	____
4	____
5	____

☐ b. Plot the points.



☐ c. Sketch the function.

LINEAR FUNCTIONS

Summary

Linear Functions

A linear function is a function whose graph is a straight line.

The equation of a linear function may be written in the form:

$y = Ax + B$ or $f(x) = Ax + B$

For example, the following functions are linear functions:

$y = x$ or $f(x) = x$

$y = 3x + 1$ or $f(x) = 3x + 1$

$y = -2x + 5$ or $f(x) = -2x + 5$

These functions are **not** linear functions:

$y = x^2 + 6$ or $f(x) = x^2 + 6$

$y = \frac{1}{x}$ or $f(x) = \frac{1}{x}$

The output value, y, **depends** on the input value, x, so y is called the **dependent** variable. Because you pick the number x (**independent** of anything else) it is called the **independent** variable.

Domain and Range of a Linear Function

For any linear function $y = Ax + B$, $A \neq 0$:

- The domain is all real numbers.
- The range is all real numbers.

Graphs of Linear Functions

When you have two or more linear functions of the form $y = Ax + B$ with the same coefficient A of x, the graphs of the functions look almost the same. The lines have the same steepness but are shifted up or down depending on the value of the constant, B.

For example, look at the graphs of these functions in Figure 11.1.2:

$y = 4x$

$y = 4x + 2$

$y = 4x - 3$

Notice:

- The line $y = 4x + 2$ is the line $y = 4x$ shifted up 2 units.
- The line $y = 4x - 3$ is the line $y = 4x$ shifted down 3 units.

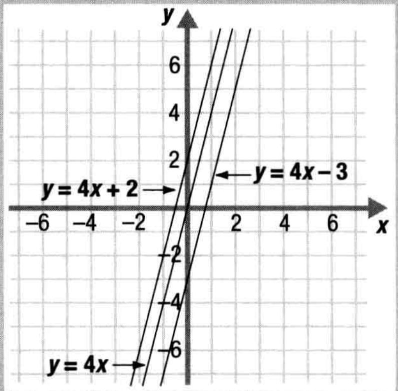


Figure 11.1.2

Here the coefficient of x in each function is 4.

Graphing Linear Functions

If you know the equation that describes a function, you can use the equation to graph the function.

To graph a function of the form $f(x) = Ax + B$:

1. Find several ordered pairs that satisfy the function.
2. Plot these points on a grid.
3. Use these points to sketch the graph of the function.

For example, to graph the function $y = f(x) = 2x + 1$:

1. Pick several values for x and find the corresponding values for $f(x)$.

x	$f(x)$
–3	–5
–1	–1
0	1
2	5

2. Plot the ordered pairs and draw a line through them. See Figure 11.1.3.

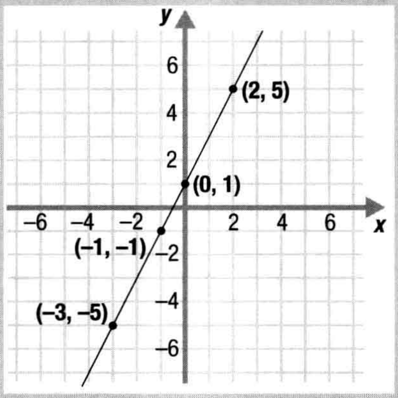


Figure 11.1.3

Absolute Value Functions

Here are four absolute value functions that are related to the linear function $y = Ax + B$:

$y = |Ax| + B$

$y = -|Ax| + B$

$y = |Ax + B|$

$y = -|Ax + B|$

The graphs of these absolute value functions are in the shape of a "V" that is either upright or inverted.

If there is a negative sign in front of the absolute value then the "V" is inverted.

Domain and Range of Absolute Value Functions

For an absolute value function based on a linear function:

- The domain is all real numbers.
- For functions of the form $y = |Ax| + B$ the range is real numbers $\geq B$.
- For functions of the form $y = -|Ax| + B$ the range is real numbers $\leq B$.
- For functions of the form $y = |Ax + B|$ the range is real numbers ≥ 0.
- For functions of the form $y = -|Ax + B|$ the range is real numbers ≤ 0.

For example, the domain and range of each of the functions below are:

$y = \|3x\| + 1$	domain: all real numbers range: real numbers ≥ 1
$y = -\|3x\| + 1$	domain: all real numbers range: real numbers ≤ 1
$y = \|3x + 1\|$	domain: all real numbers range: real numbers ≥ 0
$y = -\|3x + 1\|$	domain: all real numbers range: real numbers ≤ 0

Graphs of Absolute Value Functions

When you have two or more absolute value functions each based on a linear function and each having the same coefficient of x with no leading negative sign, their graphs look almost the same. Their only difference is the "V's" are shifted up or down depending on the constant term.

For example, look at the graphs of these absolute value functions in Figure 11.1.4:

$y = |x|$

$y = |x| + 2$

$y = |x| - 3$

Notice:

- The graph of $y = |x| + 2$ is the graph of $y = |x|$ shifted up 2 units.
- The graph of $y = |x| - 3$ is the graph of $y = |x|$ shifted down 3 units.

When there is a negative sign in front of the absolute value sign, the rules above still hold but the graphs of the functions are inverted "V's" rather than upright "V's."

For example, look at the graphs of the functions in Figure 11.1.5:

$y = -|x|$

$y = -|x| + 2$

$y = -|x| - 3$

Notice:

- The graph of $y = -|x| + 2$ is the graph of $y = -|x|$ shifted up 2 units.
- The graph of $y = -|x| - 3$ is the graph of $y = -|x|$ shifted down 3 units.

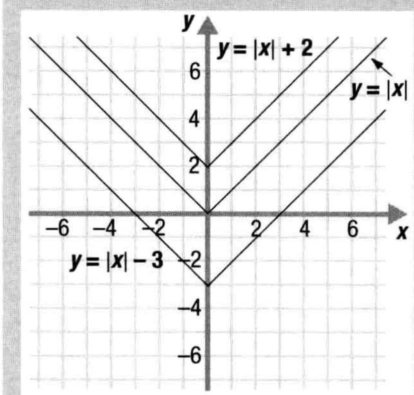


Figure 11.1.4

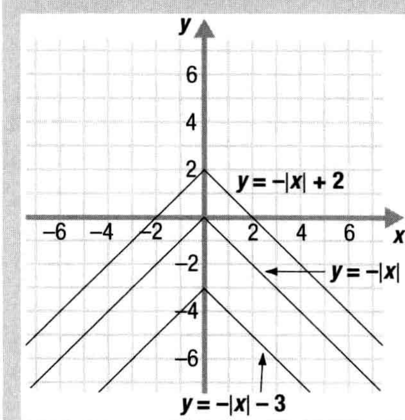


Figure 11.1.5

Answers to Sample Problems

b. all real numbers

all real numbers greater than or equal to 0 ($y \geq 0$)

c. all real numbers

all real numbers

b. $y = -x + 1$

c. $y = -x - 2$

Sample Problems

1. Find the domain and range of each of the functions below.

   a. $y = x - 1$

   b. $y = |x + 2|$

   c. $y = 5 - 3x$

   ✔ a. The domain is all the possible values for x. domain: all real numbers

   The range is all the possible values for y. range: all real numbers

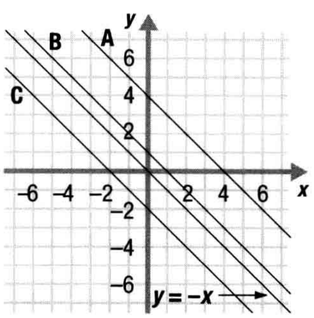
   ☐ b. The domain is all the possible values for x. domain: ______________

   The range is all the possible values for y. range: ______________

   ☐ c. The domain is all the possible values for x. domain: ______________

   The range is all the possible values for y. range: ______________

2. The graph of the function $y = -x$ is shown below. Determine the equations of the other lines shown.



   ✔ a. Find the equation of line A. This line is shifted **up** 4 units, so its equation is $y = -x + 4$.

   ☐ b. Find the equation of line B. The equation is: __________

   ☐ c. Find the equation of line C. The equation is: __________

QUADRATIC FUNCTIONS

Summary

Quadratic Functions

A quadratic function is a function whose graph is a parabola.

Equations of Quadratic Functions

The equation of a quadratic function may be written in the form:

$$y = Ax^2 + Bx + C$$

or $f(x) = Ax^2 + Bx + C$

where $A \neq 0$.

For example, the following functions are quadratic functions:

$y = 4x^2 + 5$ $\qquad$ $y = 4x^2 + 0x + 5$

$f(x) = -3x^2 + 2x - 8$ $\qquad$ $f(x) = -3x^2 + 2x + (-8)$

Notice that in each case the coefficient of the x^2-term is nonzero.

These functions are **not** quadratic functions:

$$y = \frac{4}{x^2}$$

$$y = x + 7$$

$$y = x^3 + 5x^2 - 2$$

The Graph of $y = Ax^2 + Bx + C$ when $A > 0$

The graph of a quadratic function is a parabola that opens up or down depending on the coefficient of the x^2-term. If the coefficient of the x^2-term is positive, the parabola opens up.

For example, look at the graphs of these parabolas with positive x^2-terms. See Figure 11.1.6:

$$y = x^2$$

$$y = x^2 + 2$$

$$y = x^2 - 3$$

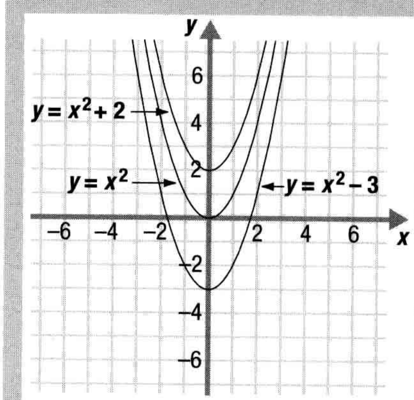


Figure 11.1.6

When the coefficient of the x^2-term is positive, the vertex of the parabola is the low point.

Notice:

- All of the graphs open up.
- The graph of $y = x^2 + 2$ is the graph of $y = x^2$ shifted up 2 units.
- The graph of $y = x^2 - 3$ is the graph of $y = x^2$ shifted down 3 units.