Mathematics for General Education

Laffer

Mathematics for General Education

WALTER B. LAFFER II
Armstrong State College

Dickenson Publishing Company, Inc., Belmont, California

Copyright © 1968 by Dickenson Publishing Company, Inc., Belmont, California. All rights reserved. No part of this book may be reproduced in any manner whatsoever without permission in writing from the publisher.

Library of Congress Catalog Card No.: 68–14852
Printed in the United States of America

Mathematics for General Education

DEDICATED TO

MARY, HARRIET, AND PAT,
who gave it the first reading

试读结束，需要全本PDF请购买 www.ertongbook.com

Preface

For some time college curricula have offered general education mathematics courses in an effort to produce a well-rounded (to use the phrasing of college catalogs) liberal arts student. Yet it does not follow that by taking one or two mathematics courses a student acquires instant roundness. And often the people who establish such requirements know little about mathematics as a twentieth-century discipline or its influence on Western culture.

This book attempts to convey a feeling for what mathematics is to a mathematician by giving the student some experience in mathematics in a general context and then exposing him to a particular aspect in depth. The "shotgun" approach to mathematics is eschewed as being not only misleading but also uninteresting. If nonmathematically inclined students are required to take mathematics, then the subject should at least be made interesting. Since mathematics *is* interesting—intellectually exciting and aesthetically pleasing—to those of us who teach it, I hope that, in this book, some of this excitement and enjoyment can be passed on to the student.

The text was class-tested at Armstrong State College, and the following formula was found to be most successful: Chapters 1 to 3 were covered during the first quarter, on four of the five class days each week. Chapter 4 or 5 was covered during the second quarter, also four days each week. On the fifth day of the week each quarter, there was a class discussion of one or two chapters of a book on the history of mathematics,* which tied in nicely with a history of Western Civilization course (usually required of college freshmen everywhere) and which made the student aware that mathematics was developed by and for people. The use of such supplementary material is extremely important for the success of the course.

If the students already have some background in mathematics, much of Chapters 1 to 3 may be used as review material. It may be advantageous to devote the entire quarter to one chapter, such as Chapter 4 or 5. The

* In this instance, *Mathematics in Western Culture* by Morris Kline (New York: Oxford University Press, 1964). Other books could serve as well; for example, *Men of Mathematics* by E. T. Bell (New York: Simon and Schuster, 1961).

student may then see theories developed and problems raised and solved, and observe some of the beauty that is internal to mathematics—for example, the neat proof. If the teacher does not share my enthusiasm for number theory and measure theory, he is encouraged to procure material for the second quarter on a topic that he likes; the bibliography lists some books on geometry, topology, the real numbers, and algebra which might be appropriate.

Chapters 1 to 3 are also appropriate for students in education. The elementary education major will find these chapters a good prerequisite for a theory of arithmetic course, and the student preparing for junior high school teaching will find that he can easily move into a course on the real number system. Although the book is not intended to replace mathematics texts written for education majors, Chapters 1 to 3 will prepare students for further study in such topics as arithmetic, the structure of the real number system, and informal geometry.

I am grateful to Dr. F. Lane Hardy, head of the Department of Mathematics, Armstrong State College, for his helpful criticism and constant encouragement; to Mr. Richard W. Hansen, president of Dickenson Publishing Company, Inc., for the same; to Professor Bevan Youse of Emory University for his fine critical reading of the manuscript; to Miss Linda J. Faircloth for her excellent job of typing and never-ending patience with my handwriting; and to Mr. Ken Williams of IBM for helping us when we were in typing trouble. Many people have contributed to the writing and completion of this book, the most patient and helpful being the Armstrong students, who have given this Yankee many valuable suggestions.

Savannah, Georgia WALTER B. LAFFER II

Mathematics for General Education

Contents

1

Logic and Sets

1.1. Truth Tables

In studying various disciplines you have no doubt discovered that you must know what is accepted as truth in the given discipline. Unless you know what truth is—or, more realistically, what the criteria are for determining whether certain statements are true in the given subject—your chance of survival in that subject is small. In some subjects, unfortunately, certain statements or phenomena are true because the professor says so. In others, the test for truth depends on our ability to transform the statement into a laboratory experiment, the truth of the statement being determined by the results of the experiment. We shall not delve into the philosophical question, "What is

试读结束，需要全本PDF请购买 www.ertongbook.com

truth?" Rather we offer a question for reflection: "Does truth mean the same to each of the following: a religious mystic, a chemist, a politician, a child absorbed in a fairy tale, a poet, a historian, a sociologist, a psychopathic killer, and a mathematician?"

The kind of truth we shall concern ourselves with is sometimes called "conditional truth." The reason is that mathematics concerns itself with conditional statements. A conditional statement is made up of at least two substatements, and its truth depends upon the truth of those substatements. For example, consider the statement, "If x is a prime number* larger than 7, then the decimal representation of x must end in a 1, 3, 7, or 9." The substatements here are "x is a prime number larger than 7" and "the decimal representation of x must end in a 1, 3, 7, or 9." These substatements are connected by the common connective form, "if . . ., then . . . ." The truth of the statement, "If x is a prime number larger than 7, then the decimal representation of x must end in a 1, 3, 7, or 9," is dependent upon the truth of the substatements.

From now on instead of using the word "substatement" we shall use the word *statement* or *proposition*, and we shall assign to every statement a truth value. We shall use lower-case letters such as $p, q, r, \ldots$ to stand for propositions. For the moment we shall not concern ourselves with the meaning of these statements. We shall allow only two possible truth values—namely, true and false. We denote these values by the letters T and F. Our interest now is in discovering the truth values for new propositions that are formed out of old propositions whose truth values are known.

At first we shall talk only about propositions that are formed by connecting *two* propositions by some connective such as "and," or "if . . ., then . . .," or "either . . . or . . . ." The truth values of propositions formed by combining more than two propositions will be derivable from the rules governing the truth values of propositions formed by combining only two propositions.

Our standard procedure will be to define the truth values of compound propositions as follows:

p	q	Something
T	T	A
T	F	B
F	T	C
F	F	D

The first two columns represent all the possible combinations of truth values of p and q together. You will note that we have (row 1) p is true and q is true, (row 2) p is true and q is false, (row 3) p is false and q is true, (row 4) p is false

* A prime number is a whole number greater than 1 that is divisible by itself and by 1 only. The first eight primes are 2, 3, 5, 7, 11, 13, 17, and 19.

and q is false. The column labeled "Something" will be in practice the new proposition. The letters A, B, C, and D will be either T's or F's. This table is called a *truth table.*

For instance, if p and q are two propositions, then we introduce the compound statement "p and q." We denote this new statement by $p \wedge q$. The wedge ($\wedge$) is our shorthand way of saying "and." We note that the truth value of the proposition $p \wedge q$ is dependent upon the truth values of the propositions p and q. In fact, there are four possibilities for the combined truth values of p and q. These are: p is true and q is true; p is true and q is false; p is false and q is true; p is false and q is false. From these four possibilities we assign corresponding truth values to the compound proposition $p \wedge q$. These would be: for p true and q true, then $p \wedge q$ is true; for p true and q false, then $p \wedge q$ is false; for p false and q true, then $p \wedge q$ is false; for p false and q false, then p $\wedge$ q is false. The values we have assigned to $p \wedge q$ are in accord with ordinary everyday usage. The compound sentence "p and q" is true only when both p and q are true.

The analysis just given can be shortened and represented in a truth table as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \wedge q$ is called the *conjunction* of p and q. *The truth table above defines the symbol* $p \wedge q$ *for us.*

Before defining further combinations of two propositions, we shall talk about one very important concept: negation. The symbol "$\sim p$" means the *negation* of the proposition p. We read "$\sim p$" as "not p."

We define $\sim p$ in the following way:

p	$\sim p$
T	F
F	T

When p is true, then $\sim p$ is false, and when p is false, then $\sim p$ is true.

The following combinations of propositions as well as the definitions above should be learned so that we can use them later on.

We read "$p \vee q$" as "p or q"; this is called the *disjunction* of p and q and is defined by the following table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

We read "$p \to q$" as "p implies q" or "if p, then q"; this is called the *conditional* and is defined by the table below:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Suppose p is the statement, "x is greater than 12," and q is the statement, "y is less than 3." Then $p \vee q$ is the compound statement, "Either x is greater than 12, or y is less than 3." In compliance with ordinary usage, certainly, this compound sentence is false only when x is less than or equal to 12 and y is greater than or equal to 3. In all other cases involving x and y the compound sentence is true.

For $p \to q$ we have, using the statements above for p and for q, "If x is greater than 12, then y is less than 3." We see that this compound sentence is false only when x is greater than 12, and y is greater than or equal to 3—that is, x is greater than 12, and y is *not* less than 3. This use of the connective, "If . . ., then . . .," is in accord with ordinary usage.

The last two rows of the truth table for $p \to q$ are probably not in accord with common usage. The reason is that in common usage we do not consider these last two possibilities. Since we are defining $p \to q$ by means of this truth table, we must complete the table to complete the definition, and it is customary to complete the table in the manner shown above. What these two rows are saying intuitively is that from a false hypothesis if we deduce a true statement then the *deduction* was valid or if we deduce a false statement then the *deduction* was valid.

A scientist who is interested in testing his hypotheses and theories by comparing results obtained by inference with results obtained by experiment always assumes that his inferences are valid. Thus if experimental results disagree with theory, he is forced to use the fourth row of the $p \to q$ truth table and thus admit that his hypothesis is false. He then, hopefully, modifies his theories and starts experimenting again.

We define "$p \leftrightarrow q$" as $(p \rightarrow q) \wedge (q \rightarrow p)$. This is read "*p if and only if q*" or "*p is logically equivalent to q*." From what we have done we can compute the truth table for $p \leftrightarrow q$:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

In computing the final column above we have used the definition of "$\wedge$." In the third column above let us for brevity give the name "a" to "$p \rightarrow q$" and in the fourth column the name "b" to "$q \rightarrow p$." Then the third, fourth, and fifth columns become:

a	b	$a \wedge b$
T	*T*	*T*
F	*T*	*F*
T	*F*	*F*
T	*T*	*T*

We have computed the truth value for $a \wedge b$ from the defining table for "$\wedge$." For instance, if a is false and b is true (second row), then $a \wedge b$ is false—and so on.

Thus, when we leave out the intermediate computational steps, our table for "$p \leftrightarrow q$" is:

p	q	$p \leftrightarrow q$
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*T*

We note that for $p \leftrightarrow q$ to be true, p and q must have the same truth value.

We shall show that "$p \rightarrow q$" and "$(\sim p) \vee q$" are logically equivalent. We do this by means of a truth table:

p	q	$\sim p$	$p \rightarrow q$	$(\sim p) \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p) \vee q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Now since "$p \rightarrow q$" and "$\sim p \vee q$" have the same truth values for the corresponding values of p and q, we may write

$$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$$

Here we note that the compound proposition "$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$" is always true. That is, no matter what truth values p and q may have, "$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$" always has the truth value T.

There are two things to observe in the demonstration above. First, we have implicitly used a substitution. That is, if we let x stand for $p \rightarrow q$ and y for $\sim p \vee q$, then in the last three columns we would have:

x	y	$x \leftrightarrow y$
T	T	T
F	F	T
T	T	T
T	T	T

This is in complete accord with the truth table for "$p \leftrightarrow q$." We have agreed that the truth table for $p \leftrightarrow q$ (and indeed for any expression) is independent of what p and q are. That is, p and q could be very complicated compound propositions or very simple propositions. Nonetheless, we have said that when two propositions p and q (or x and y) have the same truth value, then $p \leftrightarrow q$ (or $x \leftrightarrow y$) is true. We have used this idea in the previous example, where we deduced the truth table for $p \leftrightarrow q$. We used the truth table for $p \wedge q$. We let a be $p \rightarrow q$ and b be $q \rightarrow p$; then we computed the values for $a \wedge b$. We shall continue to use this process of substitution. We shall admit it as a valid process, and we shall probably not make definite mention of it again. You should be aware of this process, as it will come in very handy when you are asked to derive truth tables.

The second observation in this demonstration—that $(p \rightarrow q)$ is logically equivalent to $(\sim p \vee q)$—is that under the column $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ we have all T's. We have thus arrived at a certain combination of propositions that is "always true"—in the sense that no matter what truth values p and q have, the compound proposition has the value T. When this occurs, the compound proposition is called a *tautology*. We shall proceed to derive tautologies, for these will be our rules of inference or deduction.

Suppose we let p be the statement, "It is raining." Then $\sim p$ is the statement, "It is not raining." Now if someone were to say to you, "$p \vee \sim p$"—that is, "Either it is raining, or it is not raining"—then you would probably find it hard to believe that this someone had uttered anything very earth-shaking. Most of us would agree that this type of statement was obviously true. Our first theorem formally confirms what we already know from common usage.

THEOREM 1.1.1. $p \vee \sim p$ is a tautology.

Proof. We set up the truth table as shown. Our substitution has been to let $\sim p$ play the role of q in the truth table for $p \vee q$.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Exercises

1. Show that the following are tautologies.

   (a) $p \to p$.
   (b) $p \leftrightarrow p$.
   (c) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \to (p \leftrightarrow r)$.
   (d) $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$.
   (e) $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$.

   The tautology in (b) is called the *reflexive property of logical equivalence*. The tautology in (c) is called the *transitive property of logical equivalence*. The tautologies in (d) and (e) are called *DeMorgan's laws*.

2. Determine which of the following are tautologies. Write a truth table in each case.

   (a) $[(\sim p \vee q) \wedge p] \to q$.
   (b) $[(p \vee q) \wedge \sim q] \leftrightarrow p$.
   (c) $(p \to q) \to p$.
   (d) $p \wedge q \to q$.
   (e) $[(p \to q) \wedge (q \to r)] \to (p \to r)$.
   (f) $\sim(p \vee \sim q) \leftrightarrow \sim p \vee q$.
   (g) $(p \to q) \to q$.

1.2. Rules of Inference

We proceed to derive more tautologies. You should memorize the tautologies that have been given so far, including those in the exercises. These

试读结束，需要全本PDF请购买 www.ertongbook.com